Oblate Spheroidal Harmonics and their Applications

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II.—*Oblate Spheroidal Harmonics and their Applications.*

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§1. *Introductory.*

The harmonic functions appropriate to the oblate spheroid, which are of the form $p_n(\zeta)$, $q_n(\zeta)$, or $P_n(\iota\zeta)$, $Q_n(\iota\zeta)$, when the large letters denote the usual Legendre functions, have received but little attention.* Yet they provide, as we shall show in this memoir, a very elegant analysis of a variety of physical problems. We propose to exhibit a series of illustrations of their use, together with a large body of analysis whose applications extend very far, and lead to elegant solutions, in an analytical form, of problems which are in many cases new. In other cases—for example, the classical problems of electrified circular discs under influence—geometrical methods which lead to serious limitations have alone been effective hitherto. The analysis by spheroidal harmonics is shown to be intimately associated with that by other methods, such as the Fourier-Bessel integral method, and important theorems of analysis are involved.

We may begin with a brief summary of the more important expressions already known for these functions.† If a potential function ϕ satisfies

$$\nabla^2\phi = 0$$

and a transformation to cylindrical coordinates (z, ρ, ω) is made,

$$\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{\partial^2\phi}{\partial z^2}+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\omega^2} = 0,$$

where ρ is distance from the axis.

If we now take new co-ordinates (μ, ζ, ω) defined by

$$z = a\mu\zeta, \qquad x = a\sqrt{(1-\mu^2)(1+\zeta^2)}\cos\omega, \qquad y = a\sqrt{(1-\mu^2)(1+\zeta^2)}\sin\omega,$$

the surfaces $\mu =$ constant are the confocal hyperboloids of one sheet

$$\frac{\rho^2}{a^2(1-\mu^2)}-\frac{z^2}{a^2\mu^2} = 1,$$

and the surfaces $\zeta =$ constant are the (oblate) spheroids

$$\frac{z^2}{a^2\zeta^2}+\frac{\rho^2}{a^2(1+\zeta^2)} = 1,$$

while $\omega =$ constant gives planes. (μ, ζ, ω) are a set of orthogonal co-ordinates, and if δs_μ, δs_ζ, δs_ω are the space elements at any point,

$$\delta s_\mu = a\left(\frac{\mu^2+\zeta^2}{1-\mu^2}\right)^{\frac{1}{2}}\delta\mu, \qquad \delta s_\zeta = a\left(\frac{\mu^2+\zeta^2}{1-\zeta^2}\right)^{\frac{1}{2}}\delta\zeta, \qquad \delta s_\omega = a(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}\delta\omega;$$

* The well-known paper of C. Niven, 'Phil. Trans.,' 1886, is an outstanding exception, but has little in common with this memoir.

† For a fuller account, *vide* Lamb, 'Hydrodynamics,' Camb. Univ. Press.

while LAPLACE's equation is reducible to

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right\}+\frac{\partial}{\partial\zeta}\left\{(1+\zeta^2)\frac{\partial\phi}{\partial\zeta}\right\}=\left(\frac{1}{1+\zeta^2}-\frac{1}{1-\mu^2}\right)\frac{\partial^2\phi}{\partial\omega^2},$$

or zero, in the case of symmetry, in which a solution is given by

$$\phi=\sum_0^\infty \mathrm{P}_n(\mu)\left\{a_n p_n(\zeta)+b_n q_n(\zeta)\right\}. \quad \text{(1)}$$

a_n and b_n being any arbitrary constants, $\mathrm{P}_n(\mu)$ the ordinary zonal harmonic of degree n —it is understood that μ ranges only between ± 1, whereas ζ can range from 0 to ∞, or occasionally $-\infty$ to ∞—and $p_n(\zeta)$ is

$$p_n(\zeta)=\iota^{-n}\mathrm{P}_n(i\zeta)$$
$$=\frac{(2n)!}{2^n(n!)^2}\left\{\zeta^n+\frac{n(n-1)}{2\,.\,2n-1}\zeta^{n-2}+\frac{n(-1)(n-2)(n-3)}{2\,.\,4\,(2n-1)(2n-3)}\zeta^{n-4}+\ldots\right\}, \quad \text{(2)}$$

being finite at $\zeta=0$ and infinite at infinity, n being an integer.

The second function $q_n(\zeta)$ is similarly related to $\mathrm{Q}_n(\iota\zeta)$, and admits the formulæ

$$q_n(\zeta)=p_n(\zeta)\int_\zeta^\infty\frac{d\zeta}{(1+\zeta^2)\{p_n(\zeta)\}^2}$$
$$=(-)^n\left\{p_n(\zeta)\cot^{-1}\zeta-\frac{2n-1}{1\,.\,n}p_{n-1}(\zeta)+\frac{2n-5}{3\,.\,(n-1)}-p_{n-3}(\zeta)-\ldots\right.$$
$$=\frac{2^n(n!)^2}{(2n+1)!}\left\{\frac{1}{\zeta^{n+1}}-\frac{(n+1)(n+2)}{2\,.\,(2n+3)}\frac{1}{\zeta^{n+3}}+\frac{(n+1)(n+2)(n+3)(n+4)}{2\,.\,4\,(2n+3)(2n+5)}\frac{1}{\zeta^{n+5}}-\ldots\right\} \quad \text{(3)}$$

(if $\zeta>1$).

This function is zero at infinity.

The special spheroid $\zeta=0$ is a circular disc $z=0$, $x^2+y^2=a^2$, and the hyperboloid $\mu=0$ is the remainder of the plane $z=0$. These results are all well known. The stream function of a fluid motion given by (1), as velocity potential, becomes

$$\psi=a(1-\mu^2)(1+\zeta^2)\sum_0^\infty\frac{1}{n(n+1)}\frac{d\mathrm{P}_n}{d\mu}\left\{a_n\frac{dp_n(\zeta)}{d\zeta}+b_n\frac{dq_n(\zeta)}{d\zeta}\right\}.$$

In terms of these co-ordinates, for example, the motion of an oblate spheroid with velocity U along its axis through an infinite liquid is given by*

$$\phi=-a\mathrm{U}\mu(1-\zeta\cot^{-1}\zeta)\Big/\left(\frac{\zeta_0}{\zeta_0^2+1}-\cot^{-1}\zeta_0\right),$$
$$\psi=-\tfrac{1}{2}a^2\mathrm{U}(1-\mu^2)(1+\zeta^2)\left\{\frac{\zeta}{1+\zeta^2}-\cot^{-1}\zeta\right\}\Big/\left(\frac{\zeta_0}{1+\zeta_0^2}-\cot^{-1}\zeta_0\right),$$

where $\zeta=\zeta_0$ is the equation of the spheroid.

* LAMB, 'Hydrodynamics' (1906), p. 137.

For the circular disc, $\zeta_0 = 0$,

$$\phi = \frac{2U}{\pi}\mu a(1-\zeta\cot^{-1}\zeta) = \frac{2Ua}{\pi}P_1(\mu)\,q_1(\zeta)$$

$$\psi = \frac{Ua^2}{\pi}(1-\mu^2)(1+\zeta^2)\left\{\frac{\zeta}{1+\zeta^2}-\cot^{-1}\zeta\right\}.$$

The first is identical with the well-known integral solution ($z > 0$)*

$$\phi = -\frac{2U}{\pi}\int_0^\infty e^{-\lambda z}J_0(\lambda\rho)\frac{d}{d\lambda}\cdot\frac{\sin\lambda a}{\lambda}\cdot d\lambda,$$

and the second with

$$\psi = \frac{2U}{\pi}\rho\int_0^\infty e^{-\lambda z}J_1(\lambda\rho)\frac{d}{d\lambda}\frac{\sin\lambda a}{\lambda}d\lambda\,;$$

for it is readily seen from the relation between ϕ and ψ that any fluid motion defined by

$$\phi = \int_0^\infty e^{-\lambda z}J_0(\lambda\rho)f(\lambda)\,d\lambda$$

possesses the stream function

$$\psi = -\rho\int_0^\infty e^{-\lambda z}J_1(\lambda\rho)f(\lambda)\,d\lambda.$$

The motion of a liquid through a circular aperture of radius a in an infinite plane screen is given by

$$\phi = Ua\cot^{-1}\zeta, \qquad \psi = Ua^2\mu,$$

where U is the central velocity. These expressions are also replaced readily by corresponding definite integrals.

§2. *Further Properties of the Harmonic Functions.*

The following properties of the functions are known or at once obtainable, but it is convenient to have them grouped together here for future reference :—

$$\left.\begin{aligned}(n+1)p_{n+1}(\zeta) &= (2n+1)\zeta p_n(\zeta)+np_{n-1}(\zeta)\\ (n+1)q_{n+1}(\zeta) &= nq_{n-1}(\zeta)-(2n+1)\zeta q_n(\zeta)\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

there being a difference of sign in the two cases.

$$\left.\begin{aligned}\frac{dp_n}{d\zeta}+\frac{dp_{n-2}}{d\zeta} &= (2n-1)\,p_{n-1}, \quad (1+\zeta^2)\frac{dp_n}{d\zeta} = \frac{n(n+1)}{2n+1}(p_{n+1}+p_{n-1})\\ \frac{dq_n}{d\zeta}+\frac{dq_{n-2}}{d\zeta} &= -(2n-1)\,q_{n-1}, \quad (1+\zeta^2)\frac{dq_n}{d\zeta} = -\frac{n(n+1)}{2n+1}(q_{n+1}+q_{n-1})\end{aligned}\right\}; \quad . \quad . \quad (5)$$

* LAMB, *loc. cit.*

again with a similar difference. Moreover, these relations at once lead, when the argument is zero, to

$$p_{2n+1}(0) = 0, \quad p_{2n}(0) = \frac{(2n)!}{2^{2n}(n!)^2}. \qquad (6)$$

Also

$$q_0(\zeta) = \cot^{-1}\zeta, \quad q_1(\zeta) = 1 - \zeta\cot^{-1}\zeta, \quad q_2(\zeta) = \tfrac{1}{2}(3\zeta^2+1)\cot^{-1}\zeta - \tfrac{3}{2}\zeta.$$

And since by (4)

$$(n+1)\,q'_{n+1}(0) = nq_{n-1}(0),$$

we readily find

$$q_{2n}(0) = \frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{\pi}{2}, \quad q_{2n+1}(0) = \frac{2^{2n}(n!)^2}{(2n+1!)} \qquad (7)$$

From the other set of recurrence formulæ, after a little reduction,

$$\left(\frac{dp_{2n}}{d\zeta}\right)_0 = 0, \quad \left(\frac{dp_{2n+1}}{d\zeta}\right)_0 = \frac{(2n+1)!}{2^{2n}(n!)^2}. \qquad (8)$$

$$\left(\frac{dq_{2n}}{d\zeta}\right)_0 = -\frac{2^{2n}(n-1)!\,(n+1)!}{(2n)!}, \quad \left(\frac{dq_{2n+1}}{d\zeta}\right)_0 = -\frac{(2n+1)!}{2^{2n}(n!)^2}\cdot\frac{\pi}{2}. \qquad (9)$$

§3. *The Three-Dimensional Source of Fluid on the Axis of a Circular Obstacle.*

The theory of sources and sinks in fluid, investigated in great detail in two dimensions, has hitherto made no progress in three. The only problem for which an exact analytical solution has been obtained is that of a source in presence of a sphere, where it is well known that the image system consists of a source at the inverse point, together with a line sink. This result can be proved directly by the use of zonal harmonics, and the pressure of the source on the sphere can be determined precisely as a very simple function.

No other problem, as stated, has been similarly solved exactly, though it is not difficult in some cases to proceed by successive approximations which, however, have never suggested the exact expression.

Our present purpose is to obtain the exact expression appropriate to the case of a source on the axis of an oblate spheroidal obstacle, by the use of the spheroidal harmonic functions whose accepted theory is summarised in the last sections. It is somewhat remarkable that the problem has not been solved before, although the necessary properties of the various transcendental functions involved, more especially in relation to reductions which can be made for the circular disc, are very intricate, and have been little studied by investigators except from rather restricted and isolated points of view which do not bear on a problem of this nature. The attractive nature of the analysis bearing on the mutual relations of spheroidal, as distinct from spherical, Harmonics and the Bessel functions has never been developed in detail.

Some of the results of pure analysis relevant in this connection are foreshadowed by many formulæ of Schafheitlin, in which spherical harmonics are mainly concerned.

SCHAFHEITLIN developed his formulæ of connection between Legendre and Bessel functions for their mathematical interest, and they do not appear hitherto to have been applied to any physical problem, or even to have arisen in connection with any. It may be stated, in fact, that they are isolated parts of a very general scheme of relation of Bessel functions to spherical and spheroidal harmonics which would be worthy of further developments on its purely mathematical side. In addition to SCHAFHEITLIN, it is only necessary to refer to WHITTAKER* and MACDONALD,† who have obtained formulæ which are in a sense converse to previously known relations, but again without indication of their physical applications or connection with problems of spheroidal harmonics.

The typical SCHAFHEITLIN formula,‡ which will serve to indicate the kind of relation under notice, is—for all positive integral values of n (zero included)—

$$P_{2n}(\mu) = (-)^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty J_{2n+\frac{1}{2}}(x) \cos \mu x \, . \, x^{-\frac{1}{2}} \, dx,$$

where $\mu < 1$.

§4. *The Inverse Distance Formula.*

Let R be the distance between the point $(0, 0, -c)$ on the axis of z, and the variable point (x, y, z), or $R^2 = x^2 + y^2 + (z+c)^2 = \rho^2 + (z+c)^2$.

Then

$$\frac{1}{R} = \sqrt{\left\{\frac{1}{\rho^2 + (z+c)^2}\right\}} = \int_0^\infty e^{-\lambda(z+c)} J_0(\lambda\rho) \, d\lambda$$

when $z+c$ is positive. Writing, in the oblate spheroidal co-ordinates (μ, ζ, ω),

$$z = a\mu\zeta, \quad x = a(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}} \cos \omega, \quad y = a(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}} \sin \omega,$$

then

$$\frac{1}{R} = \frac{1}{a}\left(1-\mu^2+\zeta^2+\frac{c^2}{a^2}+2\frac{c}{a}\mu\zeta\right)^{-\frac{1}{2}}$$

$$= \int_0^\infty e^{-\lambda c - \lambda a\mu\zeta} J_0\{\lambda a \sqrt{\{(1-\mu^2)(1-\zeta^2)\}} \, d\lambda.$$

We can therefore express R^{-1} in a series of spheroidal harmonics if the function

$$e^{-\lambda a\mu\zeta} J_0\{\lambda a \sqrt{(1-\mu^2)(1+\zeta^2)}\}$$

can be so expressed. As regards the appropriate expression, the following considerations may be noticed:—

The whole fluid potential in the case of the source and a spheroidal obstacle $\zeta = \zeta_0$ consists of (1) a term ϕ due to the source, and (2) a disturbance ϕ' due to the obstacle,

* 'Proc. Lond. Math. Soc.,' vol. 35 (1903), pp. 198–206.

† MACDONALD, *ibid.*, pp. 428–443.

‡ WHITTAKER, *loc. cit.*

and vanishing at infinity, therefore requiring the functions $q_n(\zeta)$ but not $p_n(\zeta)$ in its expression. This disturbance must in fact be of the type

$$\phi' = \sum_{n=0}^{\infty} b_n \mathrm{P}_n(\mu)\, q_n(\zeta),$$

where b_n depends only on c and a. The spheroid is defined by the relation

$$\frac{z^2}{a^2\zeta^2} + \frac{\rho^2}{a^2(1+\zeta^2)} = 1.$$

The condition at the obstacle is, when $\zeta = \zeta_0$,

$$\frac{\partial}{\partial\zeta}(\phi+\phi') = 0,$$

so that ϕ must be expressed in the form

$$\phi = \sum_0^{\infty} a_n \mathrm{P}_n(\mu)\, p_n(\zeta)$$

near the obstacle. If the source is of unit strength, $\phi = \mathrm{R}^{-1}$, which contains c/a in the same manner as ζ, so that ϕ is also a harmonic series in c/a, and admits the expression

$$\phi = \sum_0^{\infty} a_n \{p_n(c/a) + \alpha_n q_n(c/a)\}\, \mathrm{P}_n(\mu)\, p_n(\zeta).$$

But ϕ must vanish when c is infinite, and therefore $p_n(c/a)$ is absent, so that

$$\phi = \frac{1}{a}\sum_0^{\infty} a_n q_n(c/a)\, \mathrm{P}_n(\mu)\, p_n(\zeta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where a_n is constant as regards $(c/a, \mu, \zeta)$, is the necessary form of expression, if convergent. This convergency obviously requires $\zeta < c/a$. If $\zeta > c/a$, its appropriate continuation is

$$\phi = \frac{1}{a}\sum_0^{\infty} a_n p_n(c/a)\, \mathrm{P}_n(\mu)\, q_n(\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

for the same value of a_n. These formulæ are continuous when $\zeta = c/a$.

This simple argument is, on physical grounds, sufficient to prove the validity of these two forms of development of ϕ. It does not, however, readily lead to a determination of the constant a_n, which will now be found by a different process.

BAUER* has given the following formula :—

$$e^{\iota kz\cos\alpha}\,\mathrm{J}_0(k\rho\sin\alpha) = \sqrt{\frac{\pi}{2kr}}\sum_0^{\infty}\iota^n\,.\,(2n+1)\,.\,\mathrm{J}_{n+\frac{1}{2}}(kr)\,\mathrm{P}_n(\cos\theta)\,\mathrm{P}_n(\cos\alpha) \quad . \quad (12)$$

where $z = r\cos\theta$, $\rho = r\sin\theta$.

* 'Münchener Sitzungsber.,' V. (1875), p. 263.

We can take this as our starting point, for it remains true, so long as the series does not diverge, when complex values are inserted for some of the variables. It can be proved directly in such circumstances, but we think it unnecessary to include this proof, which would lengthen the present investigation unduly.

Let $kr = \lambda a$, $\cos\theta = \mu$, $\cos\alpha = \xi$, which are mere changes of notation for real variables, and

$$e^{\iota\lambda a\mu\xi} J_0 \{\lambda a \sqrt{(1-\mu^2)(1-\xi^2}\} = \sqrt{\frac{\pi}{2\lambda a}} \sum_0^\infty \iota^n . (2n+1) J_{n+\frac{1}{2}}(\lambda a) P_n(\mu) P_n(\xi).$$

Write now

$$\iota\xi = -\zeta, \quad \text{or} \quad \xi = \iota\zeta,$$

and

$$e^{-\lambda a\mu\zeta} J_0 \{\lambda a \sqrt{(1-\mu^2)(1+\zeta^2)}\} = \sqrt{\frac{\pi}{2\lambda a}} \sum_0^\infty \iota^n (2n+1) J_{n+\frac{1}{2}}(\lambda a) P_n(\mu) . \iota^n p_n(\xi),$$

this series being still absolutely convergent for a small enough value of ζ, say ζ_1. Its actual magnitude is not relevant, and it is sufficient to notice that it is not less than unity. Thus, if $\zeta < \zeta_1$,

$$e^{-\lambda z} J_0(\lambda\rho) = \sqrt{\frac{\pi}{2\lambda a}} \sum_0^\infty (-)^n (2n+1) J_{n+\frac{1}{2}}(\lambda a) P_n(\mu) p_n(\zeta),$$

and we may write, under the same circumstances, integration term by term being valid,

$$\frac{1}{R} = \int_0^\infty e^{-\lambda(z+c)} J_0(\lambda\rho)\, d\lambda$$

$$= \sqrt{\frac{\pi}{2a}} \sum_0^\infty (-)^n . (2n+1) P_n(\mu) p_n(\zeta) \int_0^\infty e^{-\lambda c} J_{n+\frac{1}{2}}(\lambda a) \lambda^{-\frac{1}{2}} d\lambda. \quad . \quad . \quad (13)$$

§5. *The Value of a Definite Integral.*

HANKEL, in his posthumous paper,* proved the formula

$$\int_0^\infty e^{-\lambda c} J_{n+\frac{1}{2}}(\lambda a) \lambda^{-\frac{1}{2}} d\lambda = \frac{(\frac{1}{2}a)^{n+\frac{1}{2}} . n!}{c^{n+1}\Gamma(n+\frac{3}{2})} F\left(\frac{n+1}{2}, \frac{n+2}{2}, n+\tfrac{3}{2}, -\frac{a^2}{c^2}\right)$$

in the notation of hypergeometric functions, when n is an integer. This function can be recognised at once as proportional to our function $q_n(c/a)$. In fact, the formula may be written as

$$q_n\left(\frac{c}{a}\right) = \left(\frac{\pi a}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-\lambda c} J_{n+\frac{1}{2}}(\lambda a) \lambda^{=1} d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and is proved by direct expansion of the Bessel function, if $c > a$. The resulting hypergeometric function is then a convergent series.

* 'Math. Ann.,' VIII (1875), pp. 453–470.

It is more difficult to prove the formula when $c < a$, but it remains true, and can be proved by the development of a recurrence formula. For if the integral is denoted by I_n, we can show that

$$(n+1)\,I_{n+1} = nI_{n-1} - (2n+1)\frac{c}{a}\,I_n,$$

which is identical with the relation between the corresponding q-functions.

§6. *Further Development of the Inverse Distance Formula.*

We quote, in the inverse distance formula (13), the formula (14), and find

$$\frac{1}{R} = \frac{1}{\sqrt{\{\rho^2+(z+c^2)\}}} = \frac{1}{a}\sum_0^\infty (-)^n . (2n+1)\,q_n(c/a)\,P_n(\mu)\,p_n(\zeta), \quad . \quad . \quad (15)$$

where a is arbitrary, and (a, μ, ζ) are the definition of a set of oblate spheroidal coordinates including, as one of the spheroids for which ζ is constant, the circular disc $z = 0$, $\rho = c$, which corresponds to $\zeta = 0$, the rest of the plane $z = 0$ corresponding to $\mu = 0$.

The formula is closely related to one given by DOUGALL,* but developed in a very different manner. The reader should refer to this.

Important properties of spheroidal harmonics are inherent in this expansion. For example, if $c = ax$, where x can have any real magnitude, then, since

$$\frac{1}{R} = \frac{1}{a}\,\{1-\mu^2+\zeta^2+x^2+2\mu x\zeta\}^{-\frac{1}{2}},$$

we have on multiplication by $P_n(\mu)$ and integration,

$$p_n(\zeta)\,q_n(x) = \frac{(-)^n}{2}\int_{-1}^{1}\frac{P_n(\mu)\,d\mu}{\sqrt{1-\mu^2+\zeta^2+x^2+2\mu x\zeta}}, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

provided that ζ is less than x. If ζ is greater than x, $q_n(\zeta)\,p_n(x)$ is expressed by the same integral.

The formula

$$\frac{1}{R} = \frac{1}{a}\sum_0^\infty (-)^n (2n+1)\,q_n(c/a)\,P_n(\mu)\,p_n(\zeta)$$

is only absolutely convergent if $\frac{c}{a} > \zeta$. When $\frac{c}{a} < \zeta$, it is to be replaced by

$$\frac{1}{R} = \frac{1}{x}\sum_0^\infty (-)^n (2n+1)\,p_n(c/a)\,P_n(\mu)\,q_n(\zeta). \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

* 'Proc. Edin. Math. Soc.,' vol. XXXVII (1919), pp. 33–47.

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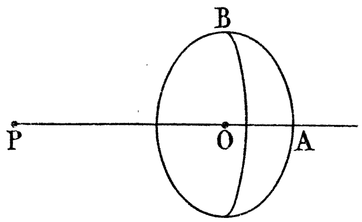
It does not seem necessary after the analogy which is evident between these formulæ and those of zonal harmonics as used for a source in presence of a sphere, to give further proof of this statement.

§7. *Source in the Presence of a Spheroid* $\zeta = \zeta_0$.

In the figure, O is the origin of co-ordinates, and the semiaxes of the spheroid are OA, OB, of lengths $a\zeta_0$, $a\sqrt{(1+\zeta_0^2)}$.



The length OP is c.

The critical case of convergency in the preceding formula, specified by $\frac{c}{a} = \zeta$, occurs on the spheroid if $c = a\zeta_0$. If P is an internal point, $\frac{c}{a} < \zeta_0$, and the formula above is used for the development of R^{-1}, the disturbance introduced by the spheroid on the liquid inside being of type

$$\phi' = \frac{1}{a}\sum_0^\infty K_n p_n(c/a) P_n(\mu) p_n(\zeta),$$

where K_n depends only on n. When P is an external point, the source of unit strength produces an effect, close to the boundary of the spheroid, given by

$$\phi = \frac{1}{a}\sum_0^\infty (-)^n (2n+1) q_n(c/a) P_n(\mu) p_n(\zeta),$$

and the spheroid produces

$$\phi' = \frac{1}{a}\sum_0^\infty (-)^n K_n (2n+1) q_n(c/a) P_n(\mu) q_n(\zeta),$$

vanishing at infinity.

Making

$$\left[\frac{\partial}{\partial\zeta}(\phi+\phi')\right]_{\zeta=\zeta_0} = 0,$$

we find

$$K_n = -p'_n(\zeta_0)/q'_n(\zeta_0),$$

and the effect of the spheroid, from which any possible image system must be deduced, is

$$\phi' = -\frac{1}{a}\sum_0^\infty (-)^n (2n+1) q_n(c/a) P_n(\mu) q_n(\zeta)\frac{p'_n(\zeta_0)}{q'_n(\zeta_0)} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

We have not been able to deduce any image system. In fact, apart from the merit of being exact, the formula is not too advantageous. It can, however, readily give useful approximations in such cases as c/a very small or very large. We do not propose to discuss such approximations in the present paper.

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§8. *Source in Front of a Circular Disc.*

When the spheroid becomes a circular disc of radius a, we have $\zeta = \zeta_0 = 0$, while $\mu = \sqrt{\left(1-\frac{\rho^2}{a^2}\right)}$ where ρ is the distance of any point of the disc from its axis. Recalling the derivates of the functions of zero argument, we find on substitution that the motion due to the disc-shaped obstacle is

$$\phi' = \frac{2}{a\pi}\sum_0^\infty (4n+3)\, q_{2n+1}(c/a)\, P_{2n+1}(\mu)\, q_{2n+1}(\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

containing only the alternate harmonics. We note that ϕ' vanishes with μ, or over the rest of the plane $z = 0$ outside the disc. This was to be expected.

The simplicity of this formula and especially the elegance of the result obtained through the disappearance of the awkward denominator $q'_n(\zeta_0)$, leads at once to the hope that this series may be summed.

§9. *Some Integral and other Properties of the Spheroidal Harmonics.*

It is well known that if m is a positive integer, and η is not a real quantity between $+1$ and -1, the formula

$$Q_m(z) = \tfrac{1}{2}\int_{-1}^{1} \frac{P_m(x)}{z-x}\, dx$$

is true universally. This has an immediate application to our present object. Writing $z = i\zeta$, where ζ is real,

$$\iota^{-m-1} q_m(\zeta) = -\tfrac{1}{2}\int_{-1}^{1} \frac{P_m(x)}{\zeta^2+x^2}(\iota\zeta+x)\, dx.$$

Let m be odd, and equal to $2n+1$, where n is a positive integer, then

$$q_{2n+1}(\zeta) = \frac{(-)^n}{2}\int_{-1}^{1} \frac{x P_{2n+1}(x)\, dx}{x^2+\zeta^2} = (-)^n \int_0^1 \frac{x P_{2n+1}(x)\, dx}{x^2+\zeta^2}, \quad . \quad . \quad . \quad . \quad . \qquad (19\text{A})$$

the imaginary part being zero by inspection.

Writing, otherwise, $m = 2n$,

$$q_{2n}(\zeta) = (-)^{n-1}\frac{i}{2}\int_{-1}^{1} \frac{P_{2n}(x)}{x^2+\zeta^2}(x+\iota\zeta)\, dx,$$

whence

$$q_{2n}(\zeta) = (-)^n \frac{\zeta}{2}\int_{-1}^{1} \frac{P_{2n}(x)}{x^2+\zeta^2}\, dx = (-)^n \zeta \int_0^1 \frac{P_{2n}(x)\, dx}{x^2+\zeta^2}. \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

These formulæ are very useful, as will appear. The last one fails when $\zeta = 0$.

Consider now a well-known formula, with (λ, μ, ω) all real,

$$P_n(\lambda\mu + \{(1-\lambda^2)(1-\mu^2)\}^{\frac{1}{2}} \cos\omega) = P_n(\lambda)\, P_n(\mu) + 2\sum_{m=1}^{\infty} \frac{n-m\,!}{n+m\,!} P_n{}^m(\lambda)\, P_n{}^m(\mu) \cos m\omega.$$

If we integrate it with regard to ω as it stands, from zero to π, as in the ordinary determination of Fourier coefficients, for this is a Fourier series, we find

$$\pi P_n(\lambda) P_n(\mu) = \int_0^\pi P_n(\lambda\mu + \{(1-\lambda^2)(1-\mu^2)\}^{\frac{1}{2}} \cos\omega)\, d\omega,$$

or if

$$\lambda\mu + \{(1-\lambda^2)(1-\mu^2)\}^{\frac{1}{2}} \cos\omega = t,$$

$$\pi P_n(\lambda) P_n(\mu) = \int \frac{P_n(t)\, dt}{\sqrt{(1-\lambda^2-\mu^2-t^2+2\lambda\mu t)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

on reduction, where the integral is taken over *all values of* t, *between* ± 1, *which make the square root real.*

Now the series, where $f(t)$ has a very general character,

$$\sum_0^\infty \tfrac{1}{2}(2n+1) P_n(t) \int_{-1}^1 f(t) P_n(t)\, dt$$

converges to $f(t)$ at all points of continuity, and to

$$\tfrac{1}{2}\{f(t-0)+f(t+0)$$

at points of discontinuity.

Suppose now that we define our function $f(t)$ by the relations

$$f(t) = (1-\lambda^2-\mu^2-t^2+2\lambda\mu t)^{-\frac{1}{2}}$$

when real,

$$f(t) = 0$$

when the square root is not real, the quantities $(\lambda\mu, t)$ being themselves all real.

Then by (21), we obtain at once, all conditions above being evidently true,

$$f(t) = \frac{\pi}{2} \sum_0^\infty (2n+1) P_n(t) P_n(\lambda) P_n(\mu),$$

a convergent development, of necessity.

The fact that, when it is real

$$(1-\lambda^2-\mu^2-t^2+2\lambda\mu t)^{-\frac{1}{2}} = \frac{\pi}{2} \sum_0^\infty (2n+1) P_n(\lambda) P_n(\mu) P_n(t) \quad . \quad . \quad . \quad . \quad (22)$$

is evidently an analogue of our inverse distance formula. It is the fundamental formula proved by DOUGALL.*

By attaching a negative sign to any one of the variables, the companion formula

$$(1-\lambda^2-\mu^2-t^2-2\lambda\mu t)^{-\frac{1}{2}} = \frac{\pi}{2} \sum_0^\infty (-)^n (2n+1) P_n(\lambda) P_n(\mu) P_n(t)$$

is obtained at once.

* *Loc. cit.*

Writing, as a completely symmetrical form in the variables,

$$f(\lambda, \mu, t) = (1-\lambda^2-\mu^2-t^2+2\lambda\mu t)^{-\frac{1}{2}} \quad \text{if real,}$$
$$= 0 \quad \text{otherwise,}$$

we find

$$f(\lambda, \mu, t) - f(\lambda, -\mu, t) = \pi \sum_0^\infty (4n+3)\, P_{2n+1}(\lambda)\, P_{2n+1}(\mu)\, P_{2n+1}(t) \quad . \quad . \quad (23)$$

resembling the series at the end of the last section, whose sum is required.

We shall also require, for future work, certain other properties of the spheroidal functions $q_n(\zeta)$. In the first place, following the lines of the usual proof, that, when $|\mu| < 1$,

$$\sum_0^\infty h^n Q_n(\mu) = (1-2h\mu+h^2)^{-\frac{1}{2}} \cosh^{-1}\left\{\frac{h-\mu}{(\mu^2-1)^{\frac{1}{2}}}\right\}$$

in the case of the Legendre function of the second kind, we can show,—it being thought unnecessary to give a proof in detail,—that

$$\sum_0^\infty k^n q_n(\zeta) = \frac{1}{(1+2k\zeta-k^2)^{\frac{1}{2}}} \cos^{-1}\frac{\zeta-k}{\sqrt{(1+\zeta^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

the series being convergent over a wide range of k or ζ. By differentiation with regard to k,

$$\sum_0^\infty nk^n q_n(\zeta) = \frac{1}{1+2k\zeta-k^2} - \frac{k(\zeta-k)}{(1+2k\zeta-k^2)^{3/2}} \cos^{-1}\frac{\zeta-k}{\sqrt{(1+\zeta^2)}},$$

changing the sign of k and subtracting,

$$2\sum_0^\infty (4n+3)\, k^{2n+1} q_{2n+1}(\zeta) = \frac{4k(1-k^2)}{(1-k^2)^2-4k^2\zeta^2} + \frac{1+k^2}{(1+2k\zeta-k^2)^{3/2}} \cos^{-1}\frac{\zeta-k}{\sqrt{(1+\zeta^2)}}$$
$$- \frac{1+k^2}{(1-2k\zeta-k^2)^{3/2}} \cos^{-1}\frac{\zeta+k}{\sqrt{(1+\zeta^2)}},$$

and writing now $k = i$, we obtain the curious result

$$\frac{1}{1+\zeta^2} = \sum_0^\infty (-)^n (4n+3)\, q_{2n+1}(\zeta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

which will be very useful. This series is absolutely convergent.

Let us now return to the preceding general theorem of DOUGALL, in the form

$$\frac{\pi}{2}\sum_0^\infty (2n+1)\, P_n(\lambda)\, P_n(\mu)\, P_n(x) = (1-\lambda^2-\mu^2-x^2+2\lambda\mu x)^{-\frac{1}{2}} \quad \text{or zero,}$$

where the first value of x, between ± 1, is also between the values making the square root real, which are

$$x = \lambda\mu \pm \sqrt{\{(1-\lambda^2)(1-\mu^2)\}},$$

We shall not lose any generality necessary for our problem if we make the restriction that λ, μ, besides being not greater than unity, are both positive. If we write $\lambda = \cos\omega_1$, $\mu = \cos\omega_2$, therefore ω_1 and ω_2 are acute, and the critical values of x are $\cos(\omega_1 \pm \omega_2)$; these values are equal to $\lambda\mu \mp \{(1-\lambda^2)(1-\mu^2)\}$, and they will be denoted by α and β $(\alpha < \beta)$. Then

$$\frac{\pi}{2}\sum_0^\infty (2n+1)\,P_n(\lambda)\,P_n(\mu)\,P_n(x) = \{(x-\alpha)(\beta-x)\}^{-\frac{1}{2}} \quad \text{or zero};$$

and if $f(x)$ be any function, finite and continuous between ± 1, but not necessarily real,

$$\frac{\pi}{2}\sum_0^\infty (2n+1)\,P_n(\lambda)\,P_n(\mu)\int_{-1}^1 P_n(x)f(x)\,dx = \int_\alpha^\beta \frac{f(x)\,dx}{\{(x-\alpha)(\beta-x)\}^{\frac{1}{2}}}.$$

Choose $f(x) = \iota\zeta - x$, then since

$$\int_{-1}^1 \frac{P_n(x)}{\iota\zeta - x}\,dx = 2\iota^{-n-1}q_n(\zeta),$$

we find

$$\pi\sum_0^\infty \iota^{-n-1}(2n+1)\,P_n(\lambda)\,P_n(\mu)\,q_n(\zeta) = \int_\alpha^\beta \frac{dx}{\iota\zeta - x}\,.\,(x-\alpha)^{-\frac{1}{2}}(\beta-x)^{-\frac{1}{2}}.$$

The integral is easily evaluated, and we find

$$\sum_0^\infty \cos(n+1)\frac{\pi}{2}\,.\,(2n+1)\,P_n(\lambda)\,P_n(\mu)\,q_n(\zeta) = -\frac{1}{\{\rho_1\rho_2\}^{\frac{1}{2}}}\cos\left(\frac{\phi_1+\phi_2}{2}\right),$$

$$\sum_0^\infty \sin(n+1)\frac{\pi}{2}\,.\,(2n+1)\,P_n(\lambda)\,P_n(\mu)\,q_n(\zeta) = +\frac{1}{\{\rho_1\rho_2\}^{\frac{1}{2}}}\sin\left(\frac{\phi_1+\phi_2}{2}\right),$$

where

$$(\phi_1, \phi_2) = \tan^{-1}\left(\frac{\zeta}{\alpha}, \frac{\zeta}{\beta}\right), \qquad \rho_1 = (\alpha^2+\zeta^2)^{\frac{1}{2}}, \qquad \rho_2 = (\beta^2+\zeta^2)^{\frac{1}{2}}.$$

These are equivalent to

$$\sum_0^\infty (-)^n(4n+3)\,P_{2n+1}(\lambda)\,P_{2n+1}(\mu)\,q_{2n+1}(\zeta) = \frac{1}{\sqrt{\rho_1\rho_2}}\cos\tfrac{1}{2}(\phi_1+\phi_2), \quad . \quad . \quad (26)$$

$$\sum_0^\infty (-)^n(4n+1)\,P_{2n}(\lambda)\,P_{2n}(\mu)\,q_{2n}(\zeta) = \frac{1}{\sqrt{\rho_1\rho_2}}\sin\tfrac{1}{2}(\phi_1+\phi_2). \quad . \quad . \quad (27)$$

The square roots must be taken positively.

For the final evaluation of the first series, which is required, we note that

$$\alpha+\beta = 2\lambda\mu, \qquad \alpha\beta = \lambda^2+\mu^2-1,$$

$$(\rho_1\rho_2)^2 = (\zeta^2+\alpha^2)(\zeta^2+\beta^2) = \zeta^4+\zeta^2(4\lambda^2\mu^2-2\lambda^2-2\mu^2+2)+(\lambda^2+\mu^2-1)^2$$

or

$$\rho_1\rho_2 = \{(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2\}^{\frac{1}{2}},$$

the *positive* value of the square root being taken in this and all other cases.

Again

$$\cos(\phi_1+\phi_2) = (\alpha\beta-\zeta^2)/\rho_1\rho_2 = 2\cos^2\frac{\phi_1+\phi_2}{2} - 1 = 1 - 2\sin^2\frac{\phi_1+\phi_2}{2},$$

and finally

$$\sum_0^\infty (-)^n (4n+3)\, P_{2n+1}(\lambda)\, P_{2n+1}(\mu)\, q_{2n+1}(\zeta)$$

$$= \frac{\left[\{(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2\}^{\frac{1}{2}} - (1+\zeta^2-\lambda^2-\mu^2)\right]^{\frac{1}{2}}}{2^{\frac{1}{2}}\{(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2\}^{\frac{1}{2}}}. \quad . \quad . \quad (28)$$

In the same manner

$$\sum_0^\infty (-)^n (4n+1)\, P_{2n}(\lambda)\, P_{2n}(\mu)\, q_{2n}(\zeta)$$

$$= \frac{\left[\{(1+\zeta^2-\lambda^2-\mu^2)+4\lambda^2\mu^2\zeta^2\}^{\frac{1}{2}} + (1+\zeta^2-\lambda^2-\mu^2)\right]^{\frac{1}{2}}}{2^{\frac{1}{2}}\{(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2\}^{\frac{1}{2}}}. \quad . \quad . \quad (29)$$

It is understood again that any square root is positive.

The presence of $q_n(\zeta)$ secures the convergency of the series. The formulæ are true when λ and μ are *positive* and do not exceed unity. It is not necessary that ζ should be real, though our proof fails if ζ is wholly imaginary between the limits $\pm i$. This case, however, will not occur in the applications, where ζ will always be real.

Some important special cases are worthy of notice. If $\lambda = 1$, we find

$$\sum_0^\infty (-)^n (4n+3)\, P_{2n+1}(\mu)\, q_{2n+1}(\zeta) = \frac{\mu}{\mu^2+\zeta^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\sum_0^\infty (-)^n (4n+1)\, P_{2n}(\mu)\, q_{2n}(\zeta) = \frac{\zeta}{\mu^2+\zeta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Further specialising by writing $\mu = 1$,

$$\sum_0^\infty (-)^n (4n+3)\, q_{2n+1}(\zeta) = \frac{1}{1+\zeta^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$\sum_0^\infty (-)^n (4n+1)\, q_{2n}(\zeta) = \frac{\zeta}{1+\zeta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The first has already been proved in this section by a much simpler method, which serves as a verification of the signs of the general results.

Any further transformation of the functions of type P_n into those of type q_n by the same method of integration term by term, serves to enhance the absolute convergency of the series, while giving valuable results for our general purposes. For example, a previous result yields

$$\sum_0^\infty (-)^n (4n+3)\, q_{2n+1}(\zeta) \int_{-1}^{1} \frac{\mu P_{2n+1}(\mu)\, d\mu}{\mu^2+\eta^2} = \int_{-1}^{1} \frac{\mu^2\, d\mu}{(\mu^2+\eta^2)(\mu^2+\zeta^2)},$$

or alternatively

$$\sum_0^\infty (4n+3)\, q_{2n+1}(\eta)\, q_{2n+1}(\zeta) = \frac{1}{\eta^2-\zeta^2}\int_0^1 d\mu \left(\frac{\eta^2}{\mu^2+\eta^2}-\frac{\zeta^2}{\mu^2+\zeta^2}\right)$$

$$= \frac{1}{\eta^2-\zeta^2}\,(\eta \cot^{-1}\eta - \zeta \cot^{-1}\zeta). \quad . \quad . \quad . \quad (34)$$

For the special case $\eta = \zeta$,

$$\sum_0^\infty (4n+3)\,[q_{2n+1}(\zeta)]^2 = \tfrac{1}{2}\left(\frac{1}{1+\zeta^2}-\frac{\cot^{-1}\zeta}{\zeta}\right). \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Similarly

$$\sum_0^\infty (-)^n (4n+1)\, q_{2n}(\zeta) \int_0^1 \frac{P_{2n}(\mu)\, d\mu}{\mu^2+\eta^2} = \zeta \int_0^1 \frac{d\mu}{(\mu^2+\eta^2)(\mu^2+\zeta^2)},$$

or

$$\sum_0^\infty (4n+1)\, q_{2n}(\eta)\, q_{2n}(\zeta) = \eta\zeta \int_0^1 d\mu \left(\frac{1}{\mu^2+\eta^2}-\frac{1}{\mu^2+\zeta^2}\right)\left(\frac{1}{\zeta^2-\eta^2}\right)$$

$$= \frac{1}{\zeta^2-\eta^2}\left\{\zeta \cot^{-1}\eta - \eta \cot^{-1}\zeta\right\}. \quad . \quad . \quad . \quad . \quad (36)$$

and in the special case $\eta = \zeta$,

$$\sum_0^\infty (4n+1)\,[q_{2n}(\zeta)]^2 = \tfrac{1}{4}\left\{\frac{\cot^{-1}\zeta}{\zeta}+\frac{1}{1+\zeta^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (37)$$

We notice that by addition

$$\frac{1}{1+\zeta^2} = \sum_0^\infty (2n+1)\,[q_n(\zeta)]^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

These formulæ all appear to be quite new. We now, in conclusion, derive the most important of all such results for the solution of our hydrodynamical problem.

Returning to (28) we have

$$\sum_0^\infty (-)^n (4n+3)\, P_{2n+1}(\lambda)\, P_{2n+1}(\mu)\, q_{2n+1}(\zeta) = \frac{\{A^{\frac{1}{2}}-(1+\zeta^2-\lambda^2-\mu^2)\}^{\frac{1}{2}}}{2^{\frac{1}{2}}A^{\frac{1}{2}}},$$

where, λ and μ being positive and less than unity,

$$A = (1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2.$$

We cannot proceed in this to the limit $\zeta = 0$ without a careful consideration of A. In $A^{\frac{1}{2}}$ the positive square root must be taken, and when $\zeta = 0$, $A = (1-\lambda^2-\mu^2)^2$. Two cases must be considered.

If

$$\lambda^2+\mu^2 < 1, \qquad A^{\frac{1}{2}} = 1-\lambda^2-\mu^2,$$

and we find

$$\sum_0^\infty (-)^n (4n+3)\, P_{2n+1}(\lambda)\, P_{2n+1}(\mu)\, q_{2n+1}(0) = 0.$$

But if

$$\lambda^2+\mu^2 > 1, \qquad A^{\frac{1}{2}} = \lambda^2+\mu^2-1,$$

and therefore equation (A)

$$\sum_0^\infty (-)^n (4n+3)\, P_{2n+1}(\lambda)\, P_{2n+1}(\mu)\, q_{2n+1}(0) = (\lambda^2+\mu^2-1)^{-\frac{1}{2}}.$$

Substituting for $\zeta_{2n+1}(0)$ by equation (7), we obtain the formula

$$\sum_0^\infty (-)^n (4n+3) \left\{\frac{2\,.\,4\,.\,6\,.\,\ldots\,2n}{1\,.\,3\,.\,5\,.\,\ldots\,2n+1}\right\} P_{2n+1}(\lambda)\, P_{2n+1}(\mu) = (\lambda^2+\mu^2-1)^{\frac{1}{2}} \quad \text{if} \quad \lambda^2+\mu^2>1$$
$$= 0 \quad \text{if} \quad \lambda^2+\mu^2<1 \quad . \quad . \quad . \quad (39)$$

λ and μ being both positive or both negative.

This is an interesting formula, in the fact that the odd numbers occur in an unusual way in the denominator. The writer has, however, obtained also another proof on different lines, which is not included in this discussion.

Similar considerations apply to functions of even order. Without giving the whole argument again, we may quote the results as follows :—

The series

$$\frac{\pi}{2}\sum_0^\infty (-)^n (4n+1) \left\{\frac{1\,.\,3\,.\,5\,.\,\ldots\,2n-1}{2\,.\,4\,.\,6\,.\,\ldots\,2n}\right\} P_{2n}(\lambda)\, P_{2n}(\mu)$$

or its equivalent

$$\sum_0^\infty (-)^n (4n+1)\, P_{2n}(\lambda)\, P_{2n}(\mu)\, q_{2n}(0)$$

has the value $(1-\lambda^2-\mu^2)^{\frac{1}{2}}$ if this quantity is real, and the value zero if not. Again, λ and μ are positive and not individually greater than unity, as in fact is evident.

This formula has also been proved by an independent method.

As suggested already, the equation (A) is one of the landmarks in the solution of our hydrodynamical problem. If we multiply by $\lambda/(\lambda^2+\eta^2)$, where η is any real quantity, and integrate from zero to unity,

$$\sum_0^\infty (-)^n (4n+3)\, P_{2n+1}(\mu)\, q_{2n+1}(0) \int_0^1 \frac{P_{2n+1}(\lambda)}{\lambda^2+\eta^2}\,.\,\lambda\, d\lambda = \int \frac{\lambda\, d\lambda}{(\lambda^2+\eta^2)\sqrt{(\lambda^2+\mu^2-1)}}$$

where the range of integration in the latter integral extends over those values between 0 and 1 for which $\lambda^2+\mu^2-1$ is positive : the range is therefore from 0 to $\sqrt{(1-\mu^2)}$. Thus

$$\sum_0^\infty (4n+3)\, P_{2n+1}(\mu)\, q_{2n+1}(0)\, q_{2n+1}(\eta) = \int_0^{\sqrt{(1-\mu^2)}} \frac{\lambda\, d\lambda}{(\lambda^2+\eta^2)\sqrt{(\lambda^2+\mu^2-1)}}$$
$$= \frac{1}{\sqrt{(1-\mu^2+\eta^2)}} \tan^{-1} \frac{\mu}{\sqrt{(1-\mu^2+\eta^2)}} \quad . \quad (40)$$

where η is real and μ positive and not greater than unity.

This formula in the next section will lead us directly to a solution of the problem in hydrodynamics, but it is interesting to continue somewhat further with these summations of series.

In particular, for instance,

$$\sum_0^\infty (4n+3)\left(\frac{2 \cdot 4 \ldots 2n}{3 \cdot 5 \ldots 2n+1}\right)^2 \mathrm{P}_{2n+1}(\mu) = \frac{\sin^{-1}\mu}{\sqrt{(1-\mu^2)}}\,(\mu<1). \quad . \quad . \quad . \quad (41)$$

whence we can express $(\sin^{-1}\mu)^2$ as a series of zonal harmonics—a new formula. By a further integration

$$\sum_0^\infty (-)^n (4n+3)\left(\frac{2 \cdot 4 \ldots 2n}{3 \cdot 5 \ldots 2n+1}\right)^3 = \int_0^1 \frac{\sin^{-1}\mu}{\sqrt{(1-\mu^2)}}\frac{d\mu}{\mu} = \int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\,dx. \quad . \quad . \quad (42)$$

In the same manner

$$\sum_0^\infty (-)^n (4n+1)\,\mathrm{P}_{2n}(\mu)\,q_{2n}(0)\,.\,\eta\int_0^1 \frac{\mathrm{P}_{2n}(\lambda)\,d\lambda}{\lambda^2+\eta^2} = \eta\int_0^{\sqrt{(1-\mu^2)}} \frac{d\lambda}{(\lambda^2+\eta^2)\sqrt{(1-\lambda^2-\mu^2)}},$$

or

$$\sum_0^\infty (4n+1)\,\mathrm{P}_{2n}(\mu)\,q_{2n}(\eta)\,q_{2n}(0) = \eta\int_0^{\frac{\pi}{2}} \frac{d\phi}{\eta^2+(1-\mu^2)\sin^2\phi},$$

or

$$\sum_0^\infty (4n+1)\,\mathrm{P}_{2n}(\mu)\,q_{2n}(\eta)\,q_{2n}(0) = \frac{\pi}{2}(1-\mu^2+\eta^2)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (43)$$

with the following consequences, which can easily be deduced,

$$\sum_0^\infty (4n+1)\frac{1 \cdot 3 \ldots 2n-1}{2 \cdot 4 \ldots 2n}\,\mathrm{P}_{2n}(\mu)\,q_{2n}(\eta) = \frac{1}{\sqrt{1-\mu^2+\eta^2}} \quad . \quad . \quad . \quad . \quad (44)$$

$$\sum_0^\infty (4n+1)\left(\frac{1 \cdot 3 \ldots 2n-1}{2 \cdot 4 \ldots 2n}\right)^2 \mathrm{P}_{2n}(\mu) = \frac{2}{\pi}\cdot\frac{1}{\sqrt{1-\mu^2}}\,(\mu \neq 1). \quad . \quad . \quad . \quad (45)$$

$$\sum_0^\infty (-)^n (4n+1)\left(\frac{1 \cdot 3 \ldots 2n-1}{2 \cdot 4 \ldots 2n}\right)^3 q_{2n}(\eta) = \frac{1}{\sqrt{1+\eta^2}}. \quad . \quad . \quad . \quad (46)$$

$$\sum_0^\infty (-)^n (4n+1)\left(\frac{1 \cdot 3 \ldots 2n-1}{2 \cdot 4 \ldots 2n}\right)^3 = \frac{2}{\pi}. \quad . \quad . \quad . \quad (47)$$

$$\sum_0^\infty (-)^n (4n+1)\left\{\frac{1 \cdot 3 \ldots 2n-1}{2 \cdot 4 \ldots 2n}\right\} q_{2n}(\eta)\,q_{2n}(\zeta) = \frac{1}{\sqrt{1+\eta^2+\zeta^2}}\tan^{-1}\left(\frac{\sqrt{1+\eta^2+\zeta^2}}{\eta\zeta}\right). \quad (48)$$

All these formulæ are of considerable utility in problems of applied mathematics arising from the circular disc, in addition to those discussed in the present paper.

§10. *Velocity Potential of Disturbance on the Disc and its Axis. Scattering of Motion.*

We showed in the penultimate section that the velocity potential due to the disc was

$$\phi' = \frac{2}{a\pi}\sum_0^\infty (4n+3)\,q_{2n+1}(c/a)\,\mathrm{P}_{2n+1}(\mu)\,q_{2n+1}(\zeta)$$

at any point (μ, ζ) in the external space. On the disc, $\zeta = 0$, and

$$(\phi')_{\zeta=0} = \frac{2}{a\pi} \sum_0^\infty (4n+3)\, q_{2n+1}(c/a)\, \mathrm{P}_{2n+1}(\mu)\, q_{2n+1}(0)$$

This series has been summed in the last section, and quoting the value

$$(\phi')_{\zeta=0} = \frac{2}{a\pi} \cdot \left(1-\mu^2+\frac{c^2}{a^2}\right)^{-\frac{1}{2}} \tan^{-1} \frac{\mu}{\sqrt{(1-\mu^2+c^2/a^2)}}.$$

The radius of the disc being a, the value of ϕ' at a point on the disc at radius ρ is, since $a\mu = \sqrt{a^2-\rho^2}$,

$$(\phi')_{\zeta=0} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{(c^2+\rho^2)}} \tan^{-1} \cdot \sqrt{\left(\frac{a^2-\rho^2}{c^2+\rho^2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

which is a comparatively simple expression. The value of ϕ on the disc, due to the unit source at the point (O, O, $-c$), is $(c^2+\rho^2)^{-\frac{1}{2}}$.

We notice, as a verification, that if the radius is infinite

$$\phi' = (c^2+\rho^2)^{-\frac{1}{2}}$$

as for a source at the image point P′, for the source is then in the presence of an infinite barrier. ϕ' is zero at $\rho = a$, and therefore is continuous, for we have seen already that its value on the rest of the plane of the disc is zero.

This continuity of ϕ at the edge of the disc is one of the most difficult conditions to satisfy analytically in such problems, if the solution is not obtained by some form of harmonic analysis. In the method of discontinuous integrals, for example, many apparent solutions of such problems can be found, in that the conditions over the disc are satisfied $(\partial\phi/\partial n)$, but it appears on trial that ϕ is not continuous at the edge, so that the suggested solution is ruled out.

The total value of the potential on the disc, including that due to the source, is

$$\phi+\phi' = \frac{1}{\sqrt{(c^2+\rho^2)}}\left\{1+\frac{2}{\pi}\tan^{-1}\sqrt{\left(\frac{a^2-\rho^2}{c^2+\rho^2}\right)}\right\} \quad . \quad . \quad . \quad . \quad . \quad (50)$$

From this fundamental formula, the entire theory can be developed. For example, in one form, the value of ϕ' at an external point with z positive is, if we use the Fourier-Bessel integral in the ordinary manner,

$$\phi' = \int_0^\infty \lambda e^{-\lambda z} \mathrm{J}_0(\lambda\rho)\, d\lambda \int_0^a \frac{2\mu\, d\mu}{\pi\sqrt{c^2+\mu^2}} \tan^{-1} \sqrt{\frac{a^2-\mu^2}{c^2+\mu^2}} \,.\, \mathrm{J}_0(\lambda\mu) \quad . \quad . \quad . \quad (51)$$

The second integral cannot apparently be evaluated in simple general terms, but if the disc is small, the integral can be readily determined to any desired order of a. This is the problem of " scattering " of the motion due to the source.

At a point on the *axis*, $\rho = 0$, and reversing the order of integration—an obviously justifiable process—

$$\phi' = \frac{2}{\pi}\int_0^a \frac{\mu\, d\mu}{\sqrt{(c^2+\mu^2)}} \tan^{-1}\sqrt{\left(\frac{a^2-\mu^2}{c^2+\mu^2}\right)} \cdot \int_0^\infty \lambda e^{-\lambda z} J_0(\lambda\mu)\, d\lambda$$

$$= -\frac{\partial}{\partial z}\cdot\frac{2}{\pi}\int_0^a \frac{\mu\, d\mu}{\sqrt{(c^2+\mu^2)(z^2+\mu^2)}} \tan^{-1}\sqrt{\left(\frac{a^2-\mu^2}{c^2+\mu^2}\right)}$$

as a single integral. At a sufficient distance, this can be expanded in negative powers of z, by writing

$$\frac{1}{\sqrt{(z^2+\mu^2)}} = \frac{1}{z}\sum_0^\infty (-)^n \frac{2n\,!}{2^{2n}(n\,!)^2}\frac{\mu^{2n}}{z^{2n}}.$$

Thus

$$\phi' = \frac{2}{\pi z^2}\sum_0^\infty \frac{(-)^n}{z^{2n}}\frac{2n+1\,!}{2^{2n}(n\,!)^2}\int_0^a \frac{\mu^{2n+1}\, d\mu}{\sqrt{c^2+\mu^2}} \tan^{-1}\sqrt{\left(\frac{a^2-\mu^2}{c^2+\mu^2}\right)}. \quad . \quad . \quad . \quad (52)$$

If this be generalised for any orientation after the usual manner, we obtain a very important formula. The zonal harmonic series due to the obstacle is

$$\phi' = \frac{2}{\pi}\sum_0^\infty (-)^n \alpha_n \cdot \frac{2n+1\,!}{2^{2n}(n\,!)^2}\frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}(\cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where

$$\alpha_n = \int_0^1 \frac{\eta^{2n+1}\, d\eta}{\sqrt{(\eta^2+c^2/a^2)}} \tan^{-1}\sqrt{\left(\frac{1-\eta^2}{\eta^2+c^2/a^2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

This expression represents the "scattering," but it ceases to be valid if $r < a$. The functions α_n are initially simple. We may also write

$$\alpha_n = \int_0^1 \frac{\eta^{2n+1}\, d\eta}{\sqrt{(\eta^2+c^2/a^2)}} \cdot \sin^{-1}\sqrt{\left(\frac{1-\eta^2}{1+(c^2/a^2)}\right)}.$$

By simple integration by parts,

$$\alpha_0 = \left[\left(\eta^2+\frac{c^2}{a^2}\right)^{\frac{1}{2}} \sin^{-1}\left(\frac{1-\eta^2}{\eta^2+c^2/a^2}\right)^{\frac{1}{2}}\right]_0^1 + \int_0^1 \frac{\eta\, d\eta}{\sqrt{(1-\eta^2)}}$$

$$= 1-\frac{c}{a}\cot^{-1}\frac{c}{a} = q_1\left(\frac{c}{a}\right).$$

This simple law, however, does not continue, and the other coefficients are not merely q functions. Thus

$$\alpha_1 = \tfrac{2}{3}\frac{c^3}{a^3}\cot^{-1}\frac{c}{a} + \tfrac{2}{3}\left(\tfrac{1}{3}-\frac{c^2}{a^2}\right).$$

They all vanish when c is infinite, as we should expect. The scattered effect at a great distance is, to a sufficient order,

$$\frac{\pi}{2}\phi' = \left(1-\frac{c}{a}\cot^{-1}\frac{c}{a}\right)\frac{a}{r^2}\cos\theta - \tfrac{1}{2}\left\{\tfrac{1}{3}-\frac{c^2}{a^2}+\frac{c^3}{a^3}\cot^{-1}\frac{c}{a}\right\}\frac{a^3}{r^4}(5\cos^3\theta - 3\cos\theta).$$

For a great distance the disc effectively acts as a doublet of strength

$$ma\left(1-\frac{c}{a}\cot^{-1}\frac{c}{a}\right),$$

where m is the strength of the source.

The function α_n admits an expansion of some elegance in powers of $(1+c^2/a^2)^{-1}$, which is very convergent, even when c and a are quite comparable magnitudes. For we may write, if $\alpha = \sqrt{(1+c^2/a^2)}$,

$$\alpha_n = \frac{1}{\alpha}\int_0^1 \eta^{2n+1}\,d\eta\,\frac{\sin^{-1} y}{\sqrt{(1-y^2)}},$$

where $y = \dfrac{\sqrt{(1-\eta^2)}}{\alpha}$ and is less than unity.

We know that

$$\frac{\sin^{-1} y}{\sqrt{(1-y^2)}} = y+\tfrac{2}{3}y^3+\frac{2\,.\,4}{3\,.\,5}y^5+\ldots = \sum_0^\infty \frac{2\,.\,4\ldots 2s}{3\,.\,5\ldots 2s+1}y^{2s+1},$$

and thus

$$\alpha_n = \frac{1}{\alpha}\sum_{s=0}^{\infty}\frac{2^{2s}(s\,!)^2}{2s+1\,!}\,\frac{1}{\alpha^{2s+1}}\int_0^1 \eta^{2n+1}(1-\eta^2)^{s+\frac{1}{2}}\,d\eta$$

$$= \frac{n\,!}{2\alpha}\sum_0^\infty \frac{2^{2s}(s\,!)^2}{(2s+1)\,!}\,\frac{\Gamma(s+\frac{3}{2})}{\Gamma(n+s+\frac{5}{2})}\,\frac{1}{\alpha^{2s+1}}$$

$$= 2^{2n+1}\frac{n\,!\;n+1\,!}{(2n+3)\,!}\cdot\frac{1}{\alpha^2}\left\{1+\frac{2}{2n+5}\cdot\frac{1}{\alpha^2}+\frac{2\,.\,4}{2n+5\,.\,2n+7}\,\frac{1}{\alpha^4}+\ldots\right\}.$$

This is the series mentioned. From it we may derive an alternative, and considerably more useful, integral for α_n. Directly from the last equation,

$$\alpha_n = 2^{2n+1}\frac{n\,!\;n+1\,!}{2n+3\,!}\,\frac{1}{\alpha^2}\,\frac{\Gamma(n+\frac{5}{2})}{\Gamma(n+\frac{3}{2})}\left\{B(n+\tfrac{3}{2},\,1)+\frac{1}{\alpha^2}B(n+\tfrac{3}{2},\,2)+\frac{1}{\alpha^4}B(n+\tfrac{3}{2},\,3)+\ldots\right\}$$

(using the ordinary B-function)

$$= 2^{2n+1}\frac{n\,!\;n+1\,!}{2n+3\,!}\,.\,(n+\tfrac{3}{2})\int_0^1 \frac{x^{n+\frac{1}{2}}\,dx}{\alpha^2-1+x}\,dx,$$

or if $x = w^2$, recalling also that $\alpha^2 = 1+c^2/a^2$,

$$\alpha_n = \frac{(n\,!)^2\,2^{2n}}{(2n+1)\,!}\int_0^1 \frac{w^{2n+2}\,dw}{w^2+c^2/a^2},$$

which is very much simpler. It shows at once that

$$\alpha_{n+1}+\frac{2n+2}{2n+3}\frac{c^2}{a^2}\alpha_n = \frac{1}{2n+3}\cdot\frac{2^{2n+2}(n+1\,!)^2}{2n+3\,!}, \quad \ldots\ldots\ldots \quad (55)$$

and with $\alpha_1 = 1-(c/a)\cot^{-1} c/a$, and this relation, the harmonic series becomes

$$\phi' = \frac{2}{\pi}\sum_0^\infty (-)^n \alpha_n \frac{2n+1\,!}{2^{2n}(n\,!)^2}\frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}(\cos\theta),$$

and can be calculated rapidly to any order of r^{-1}.

Let us now recall the formula giving the effect on the axis of z when the series is convergent. We have

$$\phi' = \frac{2}{\pi z^2}\sum_0^\infty \frac{(-)^n}{z^{2n}}\frac{2n+1\,!}{2^{2n}(n\,!)^2}a^{2n+1}\alpha_n$$
$$= \frac{2a}{\pi z^2}\int_0^1 \frac{w^2\,dw}{w^2+c^2/a^2}\left\{\sum_0^\infty (-)^n\left(\frac{wa}{z}\right)^{2n}\right\}$$

with the new value of α_n. The series involved converges absolutely and uniformly if wa/z is less than unity, or if $z > a$, and in these circumstances the processes involved are legitimate. The sum of the series is

$$1-\left(\frac{wa}{z}\right)^2+\left(\frac{wa}{z}\right)^4-\ldots = \frac{z^2}{z^2+a^2w^2},$$

and therefore

$$\phi' = \frac{2}{a\pi}\int_0^1 \frac{w^2\,dw}{(w^2+c^2/a^2)(w^2+z^2/a^2)},$$

which is symmetrical in c and z. This integral can be evaluated at once. The treatment of α_n is not valid at the critical case $c = 0$, so that this integral cannot then be relied upon.

If

$$c^2/a^2 = p^2, \qquad z^2/a^2 = q^2$$

$$\phi' = \frac{2}{a\pi}\cdot\frac{1}{p^2-q^2}\int_0^1 dw\left\{\frac{1}{w^2+q^2}-\frac{1}{w^2+p^2}\right\}$$
$$= \frac{2}{a\pi}\cdot\frac{1}{p^2-q^2}\left\{\frac{1}{q}\tan^{-1}\left(\frac{1}{q}\right)-\frac{1}{p}\tan^{-1}\frac{1}{p}\right\}$$
$$= \frac{2}{\pi}\cdot\frac{a^2}{c^2-z^2}\left\{\frac{\cot^{-1}\left(\frac{z}{a}\right)}{z}-\frac{\cot^{-1}\frac{c}{a}}{c}\right\}, \quad (56)$$

giving the potential produced on the axis by the disc. It is, of course, finite when $z = c$, as would be expected.

§11. *Pressure on the Disc.*

The potential ϕ' due to the disc, and given, as a special case from the spheroid, by

$$\phi' = \frac{2}{a\pi}\sum_0^\infty (4n+3)\, q_{2n+1}(c/a)\, P_{2n+1}(\mu)\, q_{2n+1}(\zeta)$$

at all points, is an odd function of μ. At the point $-\mu$, it becomes $-\phi'$. Corresponding points $\pm\mu$ are similarly situated at opposite sides of the disc, and their values of ϕ' are equal and opposite. In particular, the values are equal and opposite *on* the disc on its two sides. This condition is, of course, necessary to any solution.

If ϕ, on the other hand, is the potential on the disc due to the source, it is the same in sign on both sides, and the total potentials on the two sides are therefore respectively

$$\phi \pm \phi'.$$

The normal velocity is zero over the surface, and the surface velocities on the two sides are

$$\left(\frac{\partial \phi}{\partial \rho} \pm \frac{\partial \phi'}{\partial \rho}\right)_{z=0}.$$

Therefore, if σ be the density of the liquid, the resultant pressure on the disc is

$$p_1 - p_2 = \sigma \iint d\mathrm{S} \left\{ \left(\frac{\partial \phi}{\partial \rho} + \frac{\partial \phi'}{\partial \rho}\right)^2 - \left(\frac{\partial \phi}{\partial \rho} - \frac{\partial \phi'}{\partial \rho}\right)^2 \right\},$$

the integral being taken over one side of the disc. Thus

$$p_1 - p_2 = 4\sigma \iint d\mathrm{S} \frac{\partial \phi}{\partial \rho} \frac{\partial \phi'}{\partial \rho}$$

$$= 8\pi\sigma \int_0^a \frac{\partial \phi}{\partial \rho} \frac{\partial \phi'}{\partial \rho} \rho \, d\rho.$$

Now

$$(\phi')_{z=0} = \frac{2}{\pi\sqrt{(c^2-\rho^2)}} \tan^{-1} \sqrt{\left(\frac{a^2-\rho^2}{c^2+\rho^2}\right)} = \frac{2}{\pi\sqrt{(c^2+\rho^2)}} \sin^{-1} \sqrt{\left(\frac{a^2-\rho^2}{a^2+c^2}\right)},$$

$$\left(\frac{\partial \phi'}{\partial \rho}\right)_{z=0} = -\frac{2\rho}{\pi(c^2+\rho^2)} \left\{ \frac{1}{\sqrt{(c^2+\rho^2)}} \sin^{-1} \sqrt{\left(\frac{a^2-\rho^2}{a^2+c^2}\right)} + \frac{1}{\sqrt{(a^2-\rho^2)}} \right\},$$

$$(\phi)_{z=0} = \frac{1}{\sqrt{\{\rho^2+(z+c)^2\}}}, \qquad \left(\frac{\partial \phi}{\partial \rho}\right)_{z=0} = -\frac{\rho}{(\rho^2+c^2)^{3/2}},$$

and

$$p_1 - p_2 = 4\sigma \int_0^a \frac{\rho^3 d\rho}{(\rho^2+c^2)^{5/2}} \left\{ \frac{1}{\sqrt{(\rho^2+c^2)}} \sin^{-1} \sqrt{\left(\frac{a^2-\rho^2}{a^2+c^2}\right)} + \frac{1}{\sqrt{(a^2-\rho^2)}} \right\}.$$

This is the required exact expression. If the source has strength m, giving $\phi = m/\mathrm{R}$ as its potential, the result requires a factor m^2.

The evaluation of the integral is somewhat tedious, and proceeds as follows:—

Let

$$\mathrm{I}_1 = \int_0^a \frac{\rho^3 d\rho}{(\rho^2+c^2)^3} \sin^{-1} \sqrt{\left(\frac{a^2-\rho^2}{a^2+c^2}\right)},$$

$$\mathrm{I}_2 = \int_0^a \frac{\rho^3 d\rho}{(\rho^2+c^2)^{5/2}} (a^2-\rho^2)^{\frac{1}{2}},$$

so that

$$p_1-p_2 = 4\sigma m^2 (I_1+I_2).$$

Then integrating by parts,

$$I_1 = \left[\left\{-\frac{1}{\rho^2+c^2}+\frac{1}{2}\frac{c^2}{(\rho^2+c^2)^2}\right\}\sin^{-1}\sqrt{\left(\frac{a^2-\rho^2}{a^2+c^2}\right)}\right]_0^a$$

$$+\int_0^a \frac{\rho\,d\rho}{\sqrt{(a^2-\rho^2)}}\left\{-\frac{1}{(\rho^2+c^2)^{3/2}}+\frac{c^2}{2(\rho^2+c^2)^{5/2}}\right\},$$

$$= \frac{1}{2c^2}\cot^{-1}\frac{c}{a}+\left\{2\frac{\partial}{\partial(c^2)}+\frac{c^2}{2}\cdot\frac{4}{3}\left(\frac{\partial}{\partial c^2}\right)^2\right\}\int_0^a\frac{\rho\,d\rho}{\sqrt{(a^2-\rho^2)(\rho^2+c^2)}},$$

$$= \frac{1}{2c^2}\cot^{-1}\frac{c}{a}+\left\{\frac{\partial}{c\,\partial c}+\frac{c^2}{6}\left(\frac{\partial}{c\,\partial c}\right)^2\right\} I_3,$$

where

$$I_3 = \int_0^a \frac{\rho\,d\rho}{\sqrt{(a^2-\rho^2)(\rho^2+c^2)}} = \cot^{-1}\frac{c}{a}$$

by an easy reduction.

Also,

$$I_2 = \int_0^a \frac{\rho\,d\rho}{\sqrt{(a^2-\rho^2)}}\left\{\frac{1}{(\rho^2+c^2)^{3/2}}-\frac{c^2}{(\rho^2+c^2)^{5/2}}\right\}$$

$$= -\left\{\frac{\partial}{c\,\partial c}+\frac{c^2}{3}\left(\frac{\partial}{c\,\partial c}\right)^2\right\} I_3,$$

and

$$I_1+I_2 = \frac{1}{2c^2}\cot^{-1}\frac{c}{a}-\frac{c^2}{6}\left(\frac{\partial}{c\,\partial c}\right)^2 I_3$$

$$= \frac{1}{2c^2}\cot^{-1}\frac{c}{a}-\frac{c}{6}\frac{\partial}{\partial c}\cdot\frac{1}{c}\frac{\partial}{\partial c}\cot^{-1}\frac{c}{a}$$

$$= \frac{1}{2c^2}\cot^{-1}\frac{c}{a}-\frac{a}{6c}\cdot\frac{a^2+3c^2}{(a^2+c^2)^2}.$$

The pressure is therefore

$$p_1-p_2 = 2\sigma m^2\left\{\frac{1}{c^2}\cot^{-1}\frac{c}{a}-\frac{a}{6c}\cdot\frac{a^2+3c^2}{(a^2+c^2)^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

and for a distant source, becomes $\pi\sigma m^2/c^2$.

§12. *The General Solution.*

The general solution of the problem may be exhibited in various forms, but in no case can anything simpler than a definite integral be given, for though the integral can be evaluated, its expression is very long and conveys no useful information not already derived.

We may consider, in the first place, the Fourier-Bessel form of solution. Having now a complete knowledge of ϕ' over the plane $z = 0$, we may write down the general solution at once, as, on the positive side, for example,

$$\phi' = \int_0^\infty \lambda J_0(\lambda\rho)\, e^{-\lambda z}\, d\lambda \int_0^a \frac{2\mu\, d\mu}{\pi\sqrt{(c^2+\mu^2)}} J_0(\lambda\mu) \tan^{-1}\sqrt{\left(\frac{a^2-\mu^2}{c^2+\mu^2}\right)}.$$

Expanding $J_0(\lambda\mu)$ as a power series, we find

$$\int_0^a \frac{\mu\, d\mu}{\sqrt{(c^2+\mu^2)}} J_0(\lambda\mu) \tan^{-1}\sqrt{\left(\frac{a^2-\mu^2}{c^2+\mu^2}\right)} = \int_0^a \sum_{s=0}^{\infty} (-)^s \frac{\lambda^{2s}\mu^{2s+1}}{2^{2s}(s\,!)^2} \tan^{-1}\sqrt{\left(\frac{a^2-\mu^2}{c^2+\mu^2}\right)} \cdot \frac{d\mu}{\sqrt{(c^2+\mu^2)}}$$

$$= \sum_0^\infty (-)^s \frac{(\lambda a)^{2s}}{2^{2s}(s\,!)^2} \alpha_s,$$

where α_s is the integral defined in the penultimate section, where we showed that α_s was identical with a different type of integral, namely,

$$\alpha_s = \frac{2^{2s}(s\,!)^2}{2s+1\,!} \int_0^1 \frac{w^{2n+2}\, dw}{w^2+c^2/a^2}.$$

Therefore

$$\int_0^a \frac{\mu\, d\mu}{\sqrt{(c^2+\mu^2)}} J_0(\lambda\mu) \tan^{-1}\sqrt{\left(\frac{a^2-\mu^2}{c^2+\mu^2}\right)} = \int_0^1 \frac{w^2\, dw}{w^2+c^2/a^2} \sum_0^\infty (-)^s \frac{(\lambda a w)^{2s}}{2s+1\,!}$$

$$= \int_0^1 \frac{w^2\, dw}{w^2+c^2/a^2} \cdot \frac{\sin \lambda a w}{\lambda a w},$$

a very curious formula.

Finally,

$$\phi' = \frac{2}{\pi a} \int_0^\infty e^{-\lambda z} J_0(\lambda\rho)\, d\lambda \int_0^1 \frac{w \sin \lambda a w}{w^2+c^2/a^2} \cdot dw \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

(except at $c = 0$). This is the Fourier-Bessel solution.

It can be reduced to a single integral by an inversion of its order of integration, for the integral

$$\int_0^\infty \lambda e^{-\lambda z} J_0(\lambda\rho) \cos \lambda a w\, d\lambda$$

can be expressed as a function of w. Starting from the universal formula (z positive)

$$\int_0^\infty e^{-\lambda(z+\iota c)} J_0(\lambda\rho)\, d\lambda = \{\rho^2+(z+\iota c)^2\}^{-\frac{1}{2}},$$

we find, by equation of real and imaginary parts on the two sides of this relation,

$$-\int_0^\infty e^{-\lambda z} J_0(\lambda\rho) \sin \lambda c\, d\lambda = \frac{1}{\sqrt{R}} \sin \frac{\phi}{2}$$

where

$$R^2 = (\rho^2+z^2-c^2)^2+4z^2c^2, \qquad 2zc/(\rho^2+z^2-c^2) = \tan\phi,$$

and positive square roots are to be taken. The actual form is in fact discontinuous according to the relative values of (z, c, ρ). The integral required has a value somewhat complicated. The final value for ϕ' to which we are led in this way is identical with those mentioned below, though differently obtained, and we do not consider it necessary to do more than outline this method in these few words. The result is, of course, expressed in cylindrical co-ordinates (z, ρ), under the integral sign.

In the second method, we may calculate ϕ' on the axis $\rho = 0$, as in the penultimate section, where we found

$$\phi' = \frac{2}{\pi} \sum_0^\infty (-)^n \alpha_n \frac{2n+1\,!}{2^{2n}(n\,!)^2} \frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}(\cos\theta)$$

at a distance greater than a, by calculating its axial value, and generalising immediately with spherical co-ordinates. Quoting the previous value for α_n, we can then sum this series as follows:—

$$\phi' = \frac{2}{\pi} \sum_0^\infty (-)^n \frac{2n+1\,!}{2^{2n}(n\,!)^2} \frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}(\cos\theta) \cdot \frac{(n\,!)^2\, 2^{2n}}{2n+1\,!} \int_0^1 \frac{w^{2n+2}\,dw}{w^2+c^2/a^2}$$

$$= \frac{2a}{\pi r} \int_0^1 \sum_0^\infty (-)^n \left(\frac{aw}{r}\right)^{2n} P_{2n+1}(\cos\theta) \cdot \frac{w^2}{w^2+c^2/a^2}\,dw.$$

By the definition of Legendre functions as coefficients in a series,

$$\sum_0^\infty (-)^n \left(\frac{aw}{r}\right)^{2n} P_{2n+1}(\cos\theta) = \frac{1}{2i}\frac{r}{wa}\left\{\frac{1}{\sqrt{1-2hi\mu+h^2}} - \frac{1}{\sqrt{1+2hi\mu+h^2}}\right\}.$$

$$(h = a\omega/r, \qquad \mu = \cos\theta.)$$

On reduction, the expression on the right is real, and

$$\phi' = \frac{r}{\iota\pi}\int_0^1 dw \cdot \frac{w}{w^2+c^2/a^2} \cdot \left\{\frac{1}{\sqrt{r^2-a^2w^2-2\iota arw}} - \frac{1}{\sqrt{r^2-a^2w^2+2\iota arw}}\right\}, \quad . \quad (59)$$

which is the general solution in spherical co-ordinates $(r, \cos^{-1}\mu)$, on one side of the disc. Obviously we can, by virtue of the form of this expression, write down an *imaginary* image system consisting of two line distributions of sources with a definite law of density.

As stated, this integral can be evaluated, but the result is so complicated that no useful purpose is served by exhibiting the work.

Another method of procedure gives an interesting geometrical property of the general solution. We find from previous theorems

$$\frac{\pi}{2} \sum_0^\infty (2n+1) P_n(\mu) \cdot \int_{-1}^1 \int_{-1}^1 \frac{xy\,P_n(x)\,P_n(y)}{(x^2+\zeta^2)(y^2+\eta^2)}\,dx\,dy$$

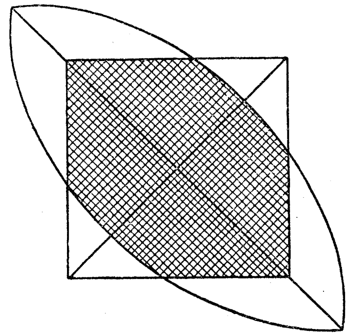
$$= \iint \frac{xy\,dx\,dy}{(x^2+\zeta^2)(y^2+\eta^2)\sqrt{1-x^2-y^2-\mu^2+2\mu xy}},$$

the latter integral being taken for values of x and y between ± 1, and making

$1-x^2-y^2-\mu^2+2\mu xy$ positive. All such values are contained within the square $x=\pm 1$, $y=\pm 1$, and the curve

$$x^2-2\mu xy+y^2 = 1-\mu^2,$$

which is an ellipse of semi-axis $\sqrt{1-\mu}$, $\sqrt{1+\mu}$ lying along the diagonals of the square. The double integral is therefore taken over the shaded area in the figure, on the left-hand side. The terms for which n is even vanish, since



$$\int_{-1}^{1} \frac{x\mathrm{P}_n(x)\,dx}{x^2+\zeta^2} = 0.$$

Thus the expression on the left may be written

$$\frac{\pi}{2}\,.\,4\,.\,\sum_0^\infty (4n+3)\,\mathrm{P}_{2n+1}(\mu)\,q_{2n+1}(\zeta)\,q_{2n+1}(\eta)\,;$$

and since

$$\phi' = \frac{2}{a\pi}\sum_0^\infty (4n+3)\,\mathrm{P}_{2n+1}(\mu)\,q_{2n+1}(c/a)\,q_{2n+1}(\zeta),$$

we find

$$\phi' = \frac{1}{a\pi^2}\iint \frac{xy\,dx\,dy}{(x^2+\zeta^2)(y^2+c^2/a^2)\sqrt{1-x^2-y^2-\mu^2+2\mu xy}} \quad . \quad . \quad . \quad . \quad (60)$$

taken over the shaded area. We do not include a proof that the result of this process is again in agreement with the formula below.

Finally, the simplest method of obtaining a single integral is to start from one of our earlier formulæ of summation of harmonic series, namely,

$$\sum_0^\infty (-)^n (4n+3)\,\mathrm{P}_{2n+1}(\mu)\,\mathrm{P}_{2n+1}(\lambda)\,q_{2n+1}(\zeta)$$
$$= \frac{\{[(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2]^{\frac{1}{2}}-(4\zeta^2-\lambda^2-\mu^2)\}^{\frac{1}{2}}}{2^{\frac{1}{2}}\{(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2\}^{\frac{1}{2}}},$$

where all square roots are taken positively, and (λ, μ) range from zero to unity.

Since

$$(-)^n\,q_{2n+1}(c/a) = \int_0^1 \frac{\lambda\mathrm{P}_{2n+1}(\lambda)}{\lambda^2+c^2/a^2}\,d\lambda,$$

we obtain immediately

$$\phi' = \frac{2}{a\pi}\int_0^1 \frac{\lambda\,d\lambda}{\lambda^2+c^2/a^2}\left\{\frac{\{[1+\zeta^2-\lambda^2-\mu^2]^2+4\lambda^2\mu^2\zeta^2\}-(1+\zeta^2-\lambda^2-\mu^2)}{2\{(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2\}}\right\}^{\frac{1}{2}}, \quad . \quad (61)$$

where the value of

$$\{(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2\}^{\frac{1}{2}}$$

requires consideration. For example, if $\zeta=0$, the value may be $1-\lambda^2-\mu^2$ or $\mu^2+\lambda^2-1$, whichever may be positive. This integral requires therefore to be divided into two parts, and its interest is mainly formal.

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The analysis of this problem is now sufficiently developed for our purpose, which, so far as the present problem is concerned, is rather to develop the requisite analysis of the subject than to elucidate the physics of one special problem in complete detail. We do not therefore examine the stream lines, whose equation follows at once from the Fourier-Bessel formula.

§13. *A Reciprocal Theorem.*

We note that the harmonic (spheroidal) series for ϕ' is symmetrical in c/a and ζ,—a fact not readily foreseen. This gives us an interesting reciprocal property. On the axis, where $\mu = 1$, $z = a\zeta$, or $\zeta = z/a$. Consider two problems concerning the same disc, one with a source at $-c$ and the other with one at $-c'$. "The effect of the disc is equal at $-c'$ in the first case and $-c$ in the second."

PART II.

§14. *Electrical Applications.*

In the present section we obtain the exact solution, by the preceding methods, of several hitherto analytically unsolved problems of electricity which are concerned with the circular disc. In each case, no exhaustive discussion of the solutions from a physical standpoint—which could be supplied at once by various modes of approximation in the majority of cases—has been included, largely from considerations of space. Such modes of approximation are usually evident, and we deem it more appropriate, for the time being, to confine ourselves closely to the mere determination of exact solutions themselves, and their verifications. When, for example, a physical quantity, such as the charge on the circular disc in various circumstances, has been expressed as a definite integral which does not admit an evaluation in terms of known functions in the general case, this integral supplies the simplest general solution possible, and its evaluation by approximate methods when certain quantities are small or large is mainly a mechanical process, and the inclusion of such processes in many cases would tend to overload the discussion and to obscure the simple general lines of argument on which all the results are based. We may state, however, that a definite integral has never been left as such, unless its evaluation by approximate methods is, in fact, a simple process.

The problems in this portion of the memoir are of a somewhat varied type, and in no sense exhaustive. We have made a selection of the more interesting of those whose solution is possible by the present methods.

§15. *A Charge $+e$ is on the Axis of an Oblate Spheroid or Circular Disc, kept at Zero Potential. To Find the Potential at any External Point, and the Charge Induced.*

If the charge is at $(0, 0, -c)$, the potential due to it alone is, in the neighbourhood of the spheroid,

$$V = \frac{E}{R} = \frac{E}{a} \sum_0^\infty (-)^n (2n+1) \, q_n(c/a) \, P_n(\mu) \, p_n(\zeta),$$

the spheroid being as before $\zeta = \zeta_0 < c/a$, or in Cartesians,

$$\frac{z^2}{a^2\zeta_0^{\ 2}} + \frac{\rho^2}{a^2(1+\zeta_0^{\ 2})} = 1,$$

and its special case, the circular disc $z = 0$, $\rho = a$. This series is convergent when ζ is nearly equal to ζ_0, or less than c/a.

The disturbance V′ due to the induced charge on the spheroid is

$$V' = \frac{E}{a} \sum_0^\infty (-)^n (2n+1)\, q_n(c/a)\, P_n(\mu)\, K_n q_n(\zeta)$$

where K_n is required. Since $V+V' = 0$ if $\zeta = \zeta_0$, $K_n = -p_n(\zeta_0)/q_n(\zeta_0)$, and thus

$$V' = -\frac{E}{a} \sum_0^\infty (-)^n (2n+1)\, q_n(c/a)\, P_n(\mu)\, q_n(\zeta) \frac{p_n(\zeta_0)}{q_n(\zeta_0)}.$$

In the special case of the circular disc, we find, with $\zeta_0 = 0$,

$$V' = -\frac{2E}{a\pi} \sum_0^\infty (4n+1)\, q_{2n}(c/a)\, P_{2n}(\mu)\, q_{2n}(\zeta). \quad . \quad . \quad . \quad . \quad . \quad (62)$$

§16. *The Induced Charge.*

When ζ is very large, it becomes r/a, where r is distance from the origin. Moreover, if ζ is large,

$$q_n(\zeta) = \frac{2^n (n\,!)^2}{2n+1\,!} \frac{1}{\zeta^{n+1}} = \frac{2^n (n\,!)^2}{2n+1\,!} \left(\frac{a}{r}\right)^{n+1},$$

and μ becomes the cosine of the spherical polar angle θ. At a sufficiently great distance, therefore, the terms of V′ are in increasing order of r^{-1}, and the first is

$$V' = -\frac{E}{a} q_0\left(\frac{c}{a}\right) \frac{p_0(\zeta_0)}{q_0(\zeta_0)} \frac{a}{r},$$

showing that the *total charge* induced on the spheroid is

$$E' = -Eq_0\left(\frac{c}{a}\right) \Big/ q_0(\zeta_0),$$

since $p_0(\zeta_0) = 1$. This very fundamental result appears to be new.

If e is the eccentricity of the generating section of the spheroid, we find

$$E' = -E \cot^{-1}\left(\frac{c}{a}\right) \Big/ \cot^{-1}(\zeta_0) = -E \cot^{-1}\left(\frac{c}{a}\right) \Big/ \sin^{-1} e. \quad . \quad . \quad . \quad (63)$$

The formula admits a variety of verifications. For example, in the case of a circular disc, it becomes

$$E' = -\frac{2E}{\pi} \cot^{-1}\left(\frac{c}{a}\right),$$

where a is the radius, and c the distance of E from the disc. For a very large disc $a = \infty$, and $E' = -E$, which is correct.

We can pass to the case of a sphere by taking a small and ζ_0 large so that $a\zeta_0 = R$, the radius of the sphere. In this case,

$$E' = -E \tan^{-1}\left(\frac{a}{c}\right) \Big/ \tan^{-1}\left(\frac{a}{R}\right) = -\frac{ER}{c}$$

in agreement with the ordinary theory.

The actual distribution of charge on the spheroid, however, does not admit a simple expression. The surface density is σ, where

$$4\pi\sigma = -\frac{\partial}{\partial n}(V+V') = -\frac{1}{a}\sqrt{\frac{1+\zeta_0^2}{\mu^2+\zeta_0^2}}\left[\frac{\partial}{\partial\zeta}(V+V')\right]_{\zeta=\zeta_0}$$

$$= -\frac{E}{a^2}\sqrt{\frac{1+\zeta_0^2}{\zeta_0^2+\mu^2}}\sum_0^\infty(-)^n(2n+1)\,q_n\left(\frac{c}{a}\right)P_n(\mu)\left\{p_n'(\zeta_0)-\frac{p_n(\zeta_0)}{q_n(\zeta_0)}q_n'(\zeta_0)\right\};$$

or since

$$p_n'(\zeta)\,q_n(\zeta)-q_n'(\zeta)\,p_n(\zeta) = \frac{1}{1+\zeta^2},$$

$$\sigma = -\frac{E}{4\pi a^2}(\zeta_0+\mu^2)^{-\frac{1}{2}}(1+\zeta_0^2)^{-\frac{1}{2}}\sum_0^\infty(-)^n(2n+1)\,q_n\left(\frac{c}{a}\right)\frac{P_n(\mu)}{q_n(\zeta_0)}.$$

If the charge E is somewhat distant, this readily yields a convergent development in power of a/c.

For the circular disc two distinct series require summation. As, however, we know V, it is sufficient to find σ_2, the portion of σ corresponding to V′, which is

$$\sigma_2 = \frac{E}{4\pi a^2\mu}\sum_0^\infty(-)^n(2n+1)\,q_n\left(\frac{c}{a}\right)P_n(\mu)\frac{p_n(0)\,q_n'(0)}{q_n(0)}$$

$$= \frac{E}{2\pi^2 a^2\mu}\sum_0^\infty(4n+1)\,q_{2n}\left(\frac{c}{a}\right)P_{2n}(\mu)\,q_{2n}'(0).$$

The direct summation of this series by our previous methods presents difficulties which make the investigation very long, for the formula

$$q_{2n}(\eta) = (-)^n\eta\int_0^1\frac{P_{2n}(x)}{x^2+\eta^2}$$

fails when $\eta = 0$. We therefore do not include the investigation, as the surface density can be obtained in an integral form by an alternative and rapid method.

§17. *Fourier-Bessel Solution for the Disc.*

On the part of the plane outside the disc, $\mu = 0$, and putting $\mu = 0$ in one of the formulæ of the second section

$$\sum_0^\infty(-)^n(4n+1)\,P_{2n}(\lambda)\,P_{2n}(\mu)\,q_{2n}(\zeta) = \left\{\frac{\{(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2\}^{\frac{1}{2}}+1+\zeta^2-\lambda^2-\mu^2}{2\{(1+\zeta^2-\lambda^2-\mu^2)^2+4\lambda^2\mu^2\zeta^2\}}\right\}^{\frac{1}{2}},$$

no doubt arising regarding the sign of square roots, as $1+\zeta^2-\mu^2$ is necessarily positive. We find

$$\sum_0^\infty (-)^n (4n+1) P_{2n}(\lambda) P_{2n}(0) q_{2n}(\zeta) = \frac{1}{\sqrt{(1+\zeta^2-\lambda^2)}} (0 \leq \lambda \leq 1). \quad . \quad . \quad (64)$$

Accordingly,

$$\sum_0^\infty (4n+1) q_{2n}(c/a) P_{2n}(0) q_{2n}(\zeta) = \frac{c}{a}\int_0^1 \frac{d\lambda}{\lambda^2+c^2/a^2}\cdot\frac{1}{\sqrt{(1+\zeta^2-\lambda^2)}}$$

$$= \frac{c}{a}\int_0^{\frac{1}{\zeta}} \frac{dt}{\frac{c^2}{a^2}+\left(1+\zeta^2+\frac{c^2}{a^2}\right)t^2} = \frac{1}{\sqrt{(1+\zeta^2+c^2/a^2)}}\tan^{-1}\left(\frac{\sqrt{(1+\zeta^2+c^2/a^2)}}{\zeta c/a}\right).$$

Accordingly,

$$(V')_{\mu=0} = \frac{-2E}{a\pi}\cdot\frac{1}{\sqrt{(1+\zeta^2+c^2/a^2)}}\tan^{-1}\left(\frac{\sqrt{(1+\zeta^2+c^2/a^2)}}{\zeta c/a}\right),$$

and when $\mu = 0$, $\rho = a\sqrt{(1+\zeta^2)}$, so that on reduction

$$(V')_{\substack{z=0\\ \rho>a}} = -\frac{2E}{\pi}\cdot\frac{1}{\sqrt{(c^2+\rho^2)}}\tan^{-1}\left\{\frac{a}{c}\sqrt{\left(\frac{c^2+\rho^2}{\rho^2-a^2}\right)}\right\}. \quad . \quad . \quad . \quad . \quad (65)$$

The value of V on this plane is $E/\sqrt{(c^2+\rho^2)}$, so that the total potential there, if $\rho > a$, is

$$V+V' = \frac{E}{\sqrt{(c^2+\rho^2)}}\left[1-\frac{2}{\pi}\tan^{-1}\left\{\frac{a}{c}\sqrt{\left(\frac{c^2+\rho^2}{\rho^2-a^2}\right)}\right\}\right], \quad . \quad . \quad . \quad . \quad (66)$$

which is zero as $\rho = a$, on the boundary of the disc. The continuity with the potential on the conductor is accordingly satisfied. We may write,

$$\left.\begin{aligned} V+V' &= \frac{2E}{\pi\sqrt{(c^2+\rho^2)}}\cot^{-1}\left\{\frac{a}{c}\sqrt{\left(\frac{c^2+\rho^2}{\rho^2-a^2}\right)}\right\} \\ &= \frac{2E}{\pi\sqrt{(c^2+\rho^2)}}\sin^{-1}\left\{\frac{c}{\rho}\sqrt{\left(\frac{\rho^2-a^2}{a^2+c^2}\right)}\right\} \qquad (\rho > a) \\ V+V' &= 0 \qquad (\rho < a) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (67)$$

and therefore, in general, the potential at any point of space (z positive) is

$$V+V' = \frac{2E}{\pi}\int_0^\infty \lambda J_0(\lambda\rho)e^{-\lambda z}d\lambda\int_a^\infty \frac{\mu J_0(\lambda\mu)}{\sqrt{(c^2+\mu^2)}}\sin^{-1}\left\{\frac{c}{\mu}\sqrt{\left(\frac{\mu^2-a^2}{a^2+c^2}\right)}\right\}d\mu. \quad . \quad (68)$$

This is the Fourier-Bessel solution of the problem. We note that if ρ is large, the potential gives the correct total charge on the disc,—a further verification.

§18. *Effect of the Disc in Spherical Harmonics.*

An expression in a series of spherical harmonics, valid at a sufficient distance, of the potential V′ due to the disc, cannot be obtained at all readily with any rigour from the

Fourier-Bessel solution. We may, however, proceed otherwise, first calculating, directly from the series in spheroidal harmonics, the effect at a distant point on the axis.

By one of the formulæ of the second section, with $\mu = 1$ (a point on the axis), we have

$$\sum_0^\infty (-)^n (4n+1) \mathrm{P}_{2n}(\lambda) \mathrm{P}_{2n}(1) q_{2n}(\zeta) = \frac{\zeta}{\zeta^2+\lambda^2},$$

there being no delicate question of sign to consider in this case, μ being a magnitude not exceeding unity. Thus with

$$q_{2n}(\eta) = (-)^n \eta \int_0^1 \frac{\mathrm{P}_{2n}(\lambda)}{\lambda^2+\eta^2} d\lambda$$

we find

$$\sum_0^\infty (4n+1) q_{2n}(\eta) \mathrm{P}_{2n}(1) q_{2n}(\zeta) = \eta\zeta \int_0^1 \frac{d\lambda}{(\lambda^2+\eta^2)(\lambda^2+\zeta^2)}.$$

The potential V′ on the axis becomes

$$\mathrm{V}' = -\frac{2\mathrm{E}}{\pi a} \sum_0^\infty (4n+1) q_{2n}(c/a) \mathrm{P}_{2n}(1) q_{2n}(\zeta)$$
$$= -\frac{2\mathrm{E}}{a\pi} \frac{c}{a} \cdot \zeta \int_0^1 \frac{d\lambda}{(\lambda^2+c^2/a^2)(\lambda^2+\zeta^2)},$$

and ζ is now z/a. Evaluating the integral, we find

$$\mathrm{V}' = -\frac{2\mathrm{E}}{a\pi} \frac{c}{a} \zeta \left(\frac{a}{c} \cot^{-1}\frac{c}{a} - \frac{1}{\zeta} \tan^{-1}\frac{1}{\zeta}\right) \Big/ \left(\zeta^2 - \frac{c^2}{a^2}\right)$$
$$= -2\mathrm{E}cz \left(\frac{1}{c} \cot^{-1}\frac{c}{a} - \frac{1}{z} \cot^{-1}\frac{z}{a}\right) \Big/ (z^2-c^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

a very simple exact expression, which, being symmetrical in z and c, shows a reciprocal property.

Instead of expanding this directly in negative powers of z, we return to the integral, which admits the development

$$\mathrm{V}' = -\frac{2\mathrm{E}}{a\pi} \frac{c}{a} \zeta \int_0^1 \left\{\frac{1}{\zeta^2} - \frac{\lambda^2}{\zeta^4} + \frac{\lambda^4}{\zeta^6} - \ldots + (-)^n \frac{\lambda^{2n}}{\zeta^{2n+2}} + \ldots\right\} \frac{d\lambda}{\lambda^2+c^2/a^2}$$

if λ exceeds unity, or z is greater than a. This becomes

$$\mathrm{V}' = -\frac{2\mathrm{E}}{\pi} \cdot \frac{c}{a} \sum_0^1 (-)^n \frac{a^{2n}}{z^{2n+1}} \int_0^1 \frac{\lambda^{2n}}{\lambda^2+c^2/a^2} d\lambda,$$

which generalises at once into the harmonic series

$$\mathrm{V}' = -\frac{2\mathrm{E}}{\pi} \cdot \frac{c}{a} \sum_0^\infty (-)^n \frac{a^{2n}}{r^{2n+1}} \mathrm{P}_{2n}(\mu) \int_0^1 \frac{\lambda^{2n}}{\lambda^2+c^2/a^2} d\lambda. \quad . \quad . \quad . \quad . \quad (70)$$

The integrals appearing in the formula are identical with those found in the last section and denoted by α_n, and their recurrence formula has been given.

It is, however, worthy of notice that the first term is

$$-\frac{2\mathrm{E}}{\pi}\cdot\frac{c}{a}\cdot\frac{1}{r}\int_0^1\frac{d\lambda}{\lambda^2+c^2/a^2}=-\frac{2\mathrm{E}}{\pi}\cdot\frac{1}{r}\cdot\cot^{-1}\frac{c}{a},$$

which is correct, since the total charge of the disc is $(-2\mathrm{E}\cot^{-1}c/a)/\pi$, as proved already. This harmonic series is valid when $r>a$.

The determination of the corresponding harmonic series for values of r less than a requires some care. On the axis, V′ is still given, of course, by the formula

$$\mathrm{V}'=-2\mathrm{E}cz\left(\frac{1}{c}\cot^{-1}\frac{c}{a}-\frac{1}{z}\cot^{-1}\frac{z}{a}\right)\Big/(z^2-c^2),$$

for which a convergent development is needed when z is small, or ζ is small, when $\zeta=z/a$, $\eta=c/a$. We may write

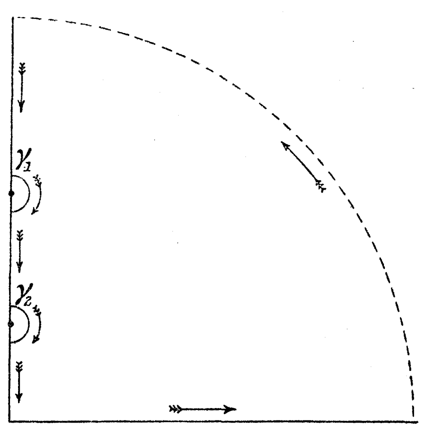
$$\frac{\eta\zeta}{\zeta^2-\eta^2}\left(\frac{1}{\eta}\tan^{-1}\frac{1}{\eta}-\frac{1}{\zeta}\tan^{-1}\frac{1}{\zeta}\right)=\frac{\eta\zeta}{\zeta^2-\eta^2}\left\{\frac{1}{\eta}\left(\frac{\pi}{2}-\tan^{-1}\eta\right)-\frac{1}{\zeta}\left(\frac{\pi}{2}-\tan^{-1}\zeta\right)\right\}$$

$$=\frac{\pi}{2(\zeta+\eta)}-\frac{\zeta\tan^{-1}\eta-\eta\tan^{-1}\zeta}{\zeta^2-\eta^2},$$

but it is not in a very convenient form for expansion in ascending powers of ζ. It is better to obtain the analytical continuation of the function by contour integration. Let the complex function of z,

$$(z^2+\eta^2)^{-1}(z^2+\zeta^2)^{-1},$$

be integrated round the contour in the figure, the small semicircles enclosing the points $z=\iota\eta$, $z=\iota\zeta$, and being of radius ε, the large quadrant being at infinity. Then the function has no poles in the contour and its integral is zero. Let x and ιy denote the real and imaginary parts of z, x and y being themselves real, and we find, since the integral on the large quadrant obviously is zero to order $1/\infty^3$,



$$\int_0^1\frac{dx}{(x^2+\eta^2)(x^2+\zeta^2)}+\int_1^\infty\frac{dx}{(x^2+\eta^2)(x^2+\zeta^2)}$$

$$+\int_{\gamma_1}+\int_{\gamma_2}+\int_3=0,$$

where γ_1 and γ_2 are the semicircles. On γ_1, we may write $z=\iota\eta+\varepsilon e^{\iota\vartheta}$, and

$$\int_{\gamma_1}=\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\frac{\iota\varepsilon e^{\iota\vartheta}\,d\vartheta}{\varepsilon e^{\iota\vartheta}\,.\,2\iota\eta\,.\,\zeta^2-\eta^2}=-\frac{\pi}{2\eta}\cdot\frac{1}{\zeta^2-\eta^2}.$$

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We assume ζ *to be positive,* so that the pole $z = \iota\zeta$ is on the upper half of the axis. Similarly,

$$\int_{\gamma_2} = -\frac{\pi}{2\zeta} \cdot \frac{1}{\eta^2-\zeta^2},$$

and also we find $\int_3$, which is the integral over the straight portion of the imaginary axis, is given in the limit $\varepsilon \to 0$ by

$$\int_3 = \mathrm{Lt}_{\epsilon\to 0}\left(\int_{+\infty}^{\eta+\epsilon} + \int_{\eta-\epsilon}^{\zeta+\epsilon} + \int_{\zeta-\epsilon}^{0}\right)\frac{\iota\, dy}{(\eta^2-y^2)(\zeta^2-y^2)} = 0$$

in the limit, being wholly imaginary before the limit. Thus

$$\int_0^1 \frac{dx}{(x^2+\eta^2)(x^2+\zeta^2)} = \frac{\pi}{2\eta\zeta(\eta+\zeta)} - \int_1^\infty \frac{dx}{(x^2+\eta^2)(x^2+\zeta^2)},$$

which is an equivalent of the previous form, suitable for convergent development in ascending powers of ζ.

We thus find

$$\begin{aligned} V' &= -\frac{2E}{a\pi}\eta\zeta\int_0^1 \frac{dx}{(x^2+\eta^2)(x^2+\zeta^2)} \\ &= -\frac{2E}{a\pi}\left\{\frac{\pi}{2(\eta+\zeta)} - \eta\zeta\int_1^\infty \frac{dx}{x^2+\eta^2}\cdot\frac{1}{x^2}\left(1-\frac{\zeta^2}{x^2}+\frac{\zeta^4}{x^4}\dots\right)\right\} \\ &= -\frac{E}{c}\left(1-\frac{z}{c}+\frac{z^2}{c^2}-\dots+(-)^n\frac{z^n}{c^n}+\dots\right) \\ &\qquad + \frac{2Ecz}{\pi a^3}\int_1^\infty \frac{dx}{x^2(x^2+\eta^2)}\left(1-\frac{z^2}{a^2x^2}+\dots+(-)^n\frac{z^{2n}}{a^{2n}x^{2n}}+\dots\right), \end{aligned}$$

and the harmonic series for points near the disc is

$$\begin{aligned} V' &= -\frac{E}{c}\left(1-\frac{r}{c}P_1(\mu)+\frac{r^2}{c^2}P_2(\mu)\dots+(-)^n\frac{r^n}{c^n}P_n(\mu)\right) \\ &\quad + \frac{2Ec}{\pi a^3}\int_1^\infty \frac{dx}{x^2(x^2+c^2/a^2)}\left(rP_1(\mu)-\frac{r^3}{a^2x^2}P_3(\mu)+\dots+(-)^n\frac{r^{2n+1}}{a^{2n}x^{2n}}P_{2n+1}(\mu)\dots\right). \quad . \quad (71) \end{aligned}$$

The two portions are of different types, the convergency of the first being determined by r/c, in relation to the charge E, and of the second by r/a, in relation to the disc. The first portion is in fact $-V$, where V is the potential due to the charge E.

In the process by which we deduced the analytical continuation of the function on the axis, we assumed ζ and therefore z to be positive.

Thus, on the positive side of the disc, with $r < a$, the total potential is given by

$$V+V' = \frac{2Ec}{\pi a^3}\int_1^\infty \frac{dx}{x^2(x^2+c^2/a^2)}\left(rP_1(\mu)-\frac{r^3}{a^2x^2}P_3(\mu)+\dots+(-)^n\frac{r^{2n+1}}{a^{2n}x^{2n}}P_{2n+1}(\mu)+\dots\right).$$

This formula must involve the "*screening*" *effect* in the region behind the disc. The series can be summed and integrated, but the result is very complicated.

Now let z, and therefore ζ, be negative. Then in the previous contour, $\int_{\gamma_2}$ is changed, since γ_2 is no longer present, but is replaced by a small semicircle round the point $-\iota\zeta$, now on the upper half of the imaginary axis. The value of the integral becomes

$$\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\iota\varepsilon e^{\iota\vartheta}\, d\vartheta}{\eta^2-\zeta^2\cdot(2i\zeta)\,\varepsilon\iota^{\iota\vartheta}} = \frac{\pi}{2\zeta}\cdot\frac{1}{(\eta^2-\zeta^2)}$$

instead of

$$-\frac{\pi}{2\eta}\cdot\frac{1}{(\zeta^2-\eta^2)}$$

and

$$\eta\zeta\int_0^1 \frac{dx}{(x^2+\eta^2)(x^2+\zeta^2)} = \frac{\pi}{2(\eta-\zeta)} - \eta\zeta\int_1^\infty \frac{dx}{x^2(x^2+\eta^2)}\left(1-\frac{\zeta^2}{x^2}+\ldots\right)$$

whence

$$V' = -\frac{E}{c}\left(1+\frac{r}{c}P_1(\mu)+\frac{r^2}{c^2}P_2(\mu)+\ldots\right)+\frac{2Ec}{\pi a^3}\int_1^\infty \frac{dx}{x^2\{x^2+(c^2/a^2)\}}\left(rP_1(\mu)-\frac{r^3}{a^2x^2}P_3(\mu)\ldots\right). \quad (72)$$

The first portion is now equal to

$$\frac{-E}{\sqrt{r^2+c^2-2rc\cos\theta}},$$

and is the effect of the ordinary "image"—E at the image point $(+c)$. The second portion then appears as the effect, on the same side of the inducing charge, of *limitation in the radius of the disc.*

We have therefore obtained harmonic series suitable for all points of space, and analysed them for the different regions in a sufficient manner for our purpose, which is to develop only a formal direct analytical treatment of the problem in all its aspects of interest.

§19. *The Surface Density.*

It is a matter of considerable difficulty to determine the surface density—with any degree of rigour—on the disc, from the general Fourier-Bessel solution.

The series just developed, however, readily lend themselves to a determination of this quantity on the two sides of the disc.

Let us consider the positive side, and determine the surface density on it corresponding to the potential V′. We have on the axis of z on this side,

$$V+V' = \frac{2Ec}{\pi a^3}\int_1^\infty \frac{dx}{x^2(x^2+c^2/a^2)}\sum_0^\infty(-)^n\frac{z^{2n+1}}{a^{2n}x^{2n}}$$

$$\frac{\partial}{\partial z}(V+V') = \frac{2Ec}{\pi a^3}\int_1^\infty \frac{dx}{x^2(x^2+c^2/a^2)}\sum_0^\infty(-)^n\frac{2^{n+1}z^{2n}}{a^{2n}x^{2n}}$$

which, as a solution of LAPLACE's equation ($r \leq a$), yields

$$\frac{\partial}{\partial z}(\mathrm{V}+\mathrm{V}') = \frac{2\mathrm{E}c}{\pi a^3}\int_1^\infty \frac{dx}{x^2(x^2+c^2/a^2)}\left\{1-\frac{3r^2}{a^2x^2}\mathrm{P}_2(\mu)+\ldots+(-)^n\frac{(2n+1)\,r^{2n}}{x^{2n}a^{2n}}\mathrm{P}_{2n}(\mu)+\ldots\right\}$$

On the positive side of the disc, where $r = \rho$, $\mu = 0$, the density is σ_2 accordingly, where

$$4\pi\sigma_2 = \frac{2\mathrm{E}c}{\pi a^3}\int_1^\infty \frac{dx}{x^2(x^2+c^2/a^2)}\left\{1+\frac{3\rho^2}{a^2x^2}\tfrac{1}{2}+\frac{5\rho^4}{a^4x^4}\cdot\frac{1\,.\,3}{2\,.\,4}+\ldots\right\}$$

$$= \frac{2\mathrm{E}c}{\pi a^3}\frac{d}{d\rho}\cdot\rho\int_1^\infty\frac{dx}{x^2(x^2+c^2/a^2)}\cdot\frac{1}{\sqrt{(1-\rho^2/a^2x^2)}}$$

$$= \frac{2\mathrm{E}c}{\pi a^2}\frac{d}{d\rho}\int_0^{\sin^{-1}\frac{\rho}{a}}\frac{\alpha^2\sin^2\phi\,d\phi}{1+\eta^2\alpha^2\sin^2\phi}$$

after a simple transformation, with $\alpha = a/\rho$, $\eta = c/a$. Thus by a simple process

$$4\pi\sigma_2 = \frac{2\mathrm{E}}{\pi c}\frac{d}{d\rho}\left\{\sin^{-1}-\frac{\rho}{a}\frac{\rho}{\sqrt{(\rho^2+c^2)}}\tan^{-1}\sqrt{\left(\frac{\rho^2+c^2}{a^2-\rho^2}\right)}\right\}$$

$$\sigma_2 = \frac{\mathrm{E}}{2\pi^2c}\cdot\frac{d}{d\rho}\left\{\sin^{-1}-\frac{\rho}{a}\frac{\rho}{\sqrt{(\rho^2+c^2)}}\sin^{-1}\sqrt{\left(\frac{\rho^2+c^2}{a^2+c^2}\right)}\right\}$$

$$= \frac{\mathrm{E}c}{2\pi^2}\left\{\frac{1}{(\rho^2+c^2)\sqrt{(a^2-\rho^2)}}-\frac{1}{(\rho^2+c^2)^{3/2}}\sin^{-1}\sqrt{\left(\frac{\rho^2+c^2}{a^2+c^2}\right)}\right\}. \quad . \quad . \quad (73)$$

This result of direct harmonic analysis agrees with KELVIN's formula obtained by the method of inversion.

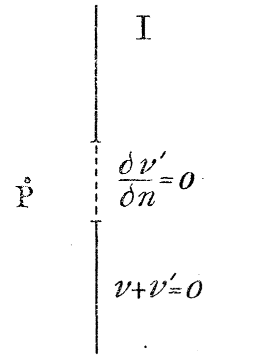
The surface density on the other side—the side on which the charge E lies—is the sum of this value and the ordinary value on an *infinite* plane due to the charge and its image, by our result of the last section. It does not seem necessary, as the two surface densities are themselves well known from the work of KELVIN, to prove that our formula $-2\mathrm{E}/\pi\cot^{-1}c/a$ is again obtained by integrating the total charge. We may, in fact, at this point take leave of the present problem as an illustration of our analysis.

§20. *Relation between Aperture and Disc Problems in Electrostatics and Hydrodynamics.*

The considerations advanced briefly in this section are not new, being in fact contained, implicitly at least, in several papers on wave motion by Lord RAYLEIGH. It seems desirable, however, to give them a compact statement, as no such statement can be found in any treatise.

Consider two problems, I and II respectively. I is that of a charge E at P, producing a potential V in space, in front of an aperture of any shape in an infinite plane earthed conductor. Let the effect of the conductor be V′, the whole potential being V+V′. Then the conditions on the plane are :—

(1) V′ is continuous at the boundary of the aperture.
(2) V+V′ = 0 over the conductor.
(3) $\partial V'/\partial n = 0$ over the aperture, by symmetry.

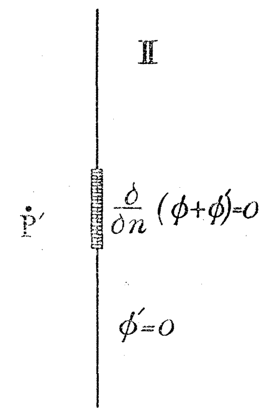


V of course necessarily satisfies (I) and need not be included.

The second problem is that of a source P′ of fluid, giving velocity potential $\phi = E/r$ at an external point, in front of an obstacle of the same shape and size, and relative position as the previous aperture, in infinite liquid.

If the effect of the obstacle is ϕ', then

(1) ϕ' is continuous (and ϕ necessarily) on the boundary.
(2) $\phi' = 0$ over the rest of the plane (corresponding to the conductor), by symmetry.
(3) $\partial/\partial n\,(\phi+\phi') - 0$ over the aperture.



If we take $\phi = V$, $\phi' = -(V+V')$, so that $\phi+\phi' = -V'$, the conditions of one problem satisfy those of the other, and the solutions are identical. We obtain, in fact, the following theorem :—

(A)—If a set of charges giving potential V is on one side of a hole in an infinite conducting wall, the whole potential being V+V′, then an exactly similar set of sources of fluid giving potential V, similarly situated with regard to an obstacle corresponding to the aperture, give a whole potential —V′.

This result may be reversed, and stated as follows :—

(B)—If a set of sources on one side of a plane obstacle in an infinite liquid gives velocity potential ϕ, and the whole potential is $\phi+\phi'$, then an exactly similar and similarly situated set of charges in front of an infinite plane conductor, with a hole corresponding to the obstacle, giving a potential ϕ, leads to a whole potential $-\phi'$ when the conductor is at zero potential.

There is also another type of correspondence, exhibited in the next figure.

In the first case, the electric charges, giving alone a potential V, are in front of a plane limited conductor. If V+V′ is the total potential, (V, V′) are continuous on the edge of the conductor, with

(1) V+V′ = 0 on the conductor.
(2) $\partial V'/\partial n = 0$, by symmetry, on the rest of the plane.

In the corresponding hydrodynamical problem, the conductor becomes an aperture in an infinite plane wall, the charges become sources, and the velocity potentials, $-\phi$.

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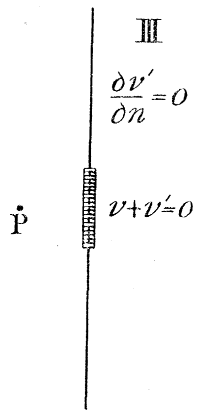
due to sources alone, ϕ' due to the barrier—satisfy, besides the conditions of continuity,

(1) $\phi' = 0$, by symmetry, over the aperture.

(2) $\partial/\partial n\,(\phi+\phi') = 0$ over the barrier.



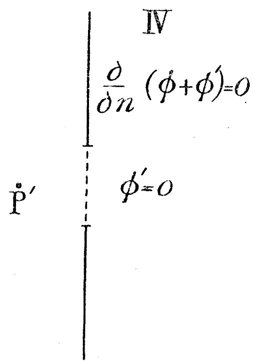
The problems exactly correspond if

$$\phi = \mathrm{V}, \qquad \phi' = -(\mathrm{V}+\mathrm{V}'), \qquad \phi+\phi' = -\mathrm{V}'$$

or the equivalents

$$\mathrm{V} = \phi, \qquad \mathrm{V}' = -(\phi+\phi'), \qquad \mathrm{V}+\mathrm{V}' = -\phi'.$$

whence the following statements :—



(C)—If a set of charges giving a potential V alone are in front of a limited plane conductor connected to earth by a fine wire, and the whole potential is V+V′, then an exactly similar and similarly situated set of sources of fluid in front of an aperture, corresponding exactly to the conductor, in an infinite plane barrier—giving a potential V themselves—give a whole potential −V′.

(D)—If a set of sources, giving alone a velocity potential ϕ in infinite liquid, are on one side of a plane infinite barrier with an aperture, with a whole potential $\phi+\phi'$ in the system, the sources may be replaced by equivalent—exactly similar and similarly situated—charges giving electric potential ϕ, and a conductor at zero potential exactly similar to the aperture, and similarly situated. The whole potential in this new system will be $-\phi'$.

At least in this explicit form, the results (A)—(D) do not appear to be widely realised, for otherwise it is inexplicable that KELVIN's solutions for the circular disc under influence should not have been hitherto translated into corresponding solutions of hydrodynamical problems of great interest, by writers on that subject. It is, of course, true that, as left by KELVIN,* the solutions relate only to induced surface density, and the corresponding development of other important physical considerations does not appear to have been carried out. Our analysis provides a ready means of carrying out some of these developments, and some attention should be paid to the subject with KELVIN's formulæ as a starting point. We have refrained from the development of the *tesseral* spheroidal harmonic analysis in this memoir, mainly because it appears necessary to work out and consolidate the simpler theory in the first place, and this alone appears to form a sufficiently large basis for the present memoir. Accordingly, we shall for

* 'Papers on Electrostatics and Magnetism,' pp. 178–191.

certain purposes avoid the use of tesseral harmonics in favour of KELVIN's surface density formulæ as a starting point.

It is to be noticed in connection with (A)—(D) that the restriction of the sources and charges to one side of the plane is not necessary, though of use in connection with brevity of statement.

The problem briefly reviewed in the next section, though closely allied to that of the circular disc under influence, does not appear to have been discussed before.

§21. *An Infinite Plane Conductor at Zero Potential contains a Circular Hole, on the Axis of which is a Charge E. To Find the Distribution on the Conductor.*

Theorem (B) of the last section leads to the solution directly from the hydrodynamical problem discussed previously, in which, corresponding to a source of potential R^{-1}, the value of ϕ' on the positive side of the plane $z=0$ was expressed, if $r>a$, in the series

$$\phi' = \frac{2}{\pi}\sum_0^\infty (-)^n \frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}(\cos\theta)\int_0^1 \frac{w^{2n+2}\,dw}{w^2+c^2/a^2}$$

the source being at $-c$.

The *whole* potential when $r>a$, z is positive, in the electrostatic problem under consideration is therefore

$$\begin{aligned} V+V' &= -E\phi' \\ &= -\frac{2E}{\pi}\sum_0^\infty (-)^n \frac{a^{2n+1}}{r^{2n+2}} P_{2n+1}(\cos\theta)\int_0^1 \frac{w^{2n+2}\,dw}{w^2+c^2/a^2}. \end{aligned}$$

There is no term in $1/r$, from which we deduce at once that the whole charge on the conductor is $-E$.

To find the surface density on the positive side, we at once derive

$$\begin{aligned} \left(\frac{\partial}{\partial z}(V+V')\right)_{z=0} &= -\frac{2E}{\pi}\sum_0^\infty (-)^{2n}(2n+2)\frac{a^{2n+1}}{\rho^{2n+3}}\frac{2n+2\,!}{2^{2n+2}(n+1\,!)^2}\int_0^1 \frac{w^{2n+2}\,dw}{w^2+c^2/a^2} \\ &= \frac{2E}{\pi a}\frac{\partial}{\partial\rho}\int_0^1 \frac{dw}{w^2+c^2/a^2}\cdot\frac{1}{\sqrt{(1-a^2w^2/\rho^2)}} \\ &= \frac{2E}{\pi}\frac{\partial}{\partial\rho}\cdot\rho\int_0^{\sin^{-1}\frac{a}{\rho}} \frac{d\lambda}{c^2+\rho^2\sin^2\lambda} \\ &= \frac{2E}{\pi}\frac{\partial}{\partial\rho}\cdot\left\{\frac{\rho}{c\sqrt{(c^2+\rho^2)}}\tan^{-1}\frac{a}{c}\sqrt{\left(\frac{\rho^2+c^2}{\rho^2-a^2}\right)}\right\} \end{aligned}$$

being infinite, as we should expect, at the edge. The density becomes

$$\sigma_2 = -\frac{E}{2\pi^2}\left\{\frac{a}{(c^2+\rho^2)\sqrt{(\rho^2-a^2)}} - \frac{c}{(c^2+\rho^2)^{3/2}}\tan^{-1}\cdot\frac{a}{c}\sqrt{\left(\frac{\rho^2+c^2}{\rho^2-a^2}\right)}\right\}$$

vanishing with a, as would be expected, for the charge on the completely infinite plane is all on the negative side, where the point charge is situated.

The quantity Q_2 of electricity on this side is

$$Q_2 = 2\pi\int_a^\infty \rho\sigma_2\, d\rho,$$

or

$$Q_2 = -\frac{E}{\pi}\int_a^\infty \rho\, d\rho\left\{\frac{a}{(c^2+\rho^2)\sqrt{(\rho^2-a^2)}} - \frac{c}{(c^2+\rho^2)^{3/2}}\tan^{-1}\frac{a}{c}\sqrt{\left(\frac{\rho^2+c^2}{\rho^2-a^2}\right)}\right\}$$

where the second term may be integrated readily by parts. The final result is

$$Q_2 = -\frac{E}{2} + \frac{Ec}{2\sqrt{(a^2+c^2)}}.$$

On the other side, there is an excess of negative electricity which can be calculated, as in our previous work, from the charge and its image in the *infinite* plane without an aperture. The excess is

$$-2\cdot\frac{Ec}{4\pi}\cdot\int_a^\infty \frac{2\pi\rho\, d\rho}{(c^2+\rho^2)^{3/2}} = -Ec/\sqrt{(a^2+c^2)}.$$

The charge on the left is therefore

$$Q_1 = -\frac{E}{2} - \frac{Ec}{2\sqrt{(a^2+c^2)}}$$

with a total charge $Q_1+Q_2 = -E$, as already deduced.

We may now take a hydrodynamical problem in illustration of the same principles.

§22. *A Source is on the Axis of a Circular Aperture in an Infinite Plane Wall in Infinite Liquid. To Find the Discharge of Liquid through the Aperture.*

Let the source give a potential $\phi = E/R$, and the wall a potential ϕ'. The discharge through the aperture in unit time is of volume

$$Q = \iint dS\left(\frac{\partial\phi}{\partial n} + \frac{\partial\phi'}{\partial n}\right)$$

taken over the aperture. The corresponding problem, by (C), is that of a charge E in front of,—at distance c, say,—a circular disc of zero potential. The potential V of the charge is ϕ. Let V′ be the potential due to the disc-conductor. Then in the hydrodynamical problem,

$$\phi+\phi' = -\mathrm{V}'$$

and

$$\mathrm{Q} = -\iint \frac{\partial \mathrm{V}'}{\partial n}\, ds.$$

This is the mean of the charges on the two sides of the disc, when taken positively, or half the total charge $-(2\mathrm{E}/\pi)\cot^{-1}(c/a)$, multiplied by 4π.

Thus

$$\mathrm{Q} = \left(\frac{\mathrm{E}}{\pi}\cot^{-1}\frac{c}{a}\right)4\pi = 4\mathrm{E}\cot^{-1}\frac{c}{a}$$

is the quantity of liquid crossing the aperture in unit time, while a volume $4\pi\mathrm{E}$ leaves the source. The fraction passing through the aperture is $1/\pi \cot^{-1} c/a$, which vanishes with the aperture when $a = 0$, with the total flow when $c = \infty$, and becomes $\frac{1}{2}\mathrm{E}$ when the source is in the aperture.

§23. *Potential of a Circular Ring in a Spheroidal Harmonic Series.*

The inverse distance formula, with $\eta = c/a$, becomes

$$\frac{1}{\mathrm{R}} = \{\rho^{\frac{1}{2}}+(z+c)^2\}^{\frac{1}{2}} = \frac{1}{a\sqrt{1-\mu^2+\zeta^2+\eta^2+2\zeta\mu\eta}}$$

$$= \sum_0^\infty (-)^n (2n+1)\, q_n(\eta)\, \mathrm{P}_n(\mu)\, p_n(\zeta) \quad (\zeta < \eta)$$

$$= \sum_0^\infty (-)^n (2n+1)\, p_n(\eta)\, \mathrm{P}_n(\mu)\, q_n(\zeta) \quad (\zeta > \eta)$$

where

$$z = a\mu\zeta, \qquad \rho = a\sqrt{(1-\mu^2)(1+\zeta^2)}.$$

In this formula, from a non-geometrical standpoint, (ζ, η, μ) may be any real quantities subject to certain restrictions.

Let now a circular ring with charge E and radius b be situated normally to the axis of z with its centre at the point $z = -c$. Its potential on the axis of z is

$$\mathrm{V} = \frac{\mathrm{E}}{\sqrt{b^2+(z+c)^2}} = \frac{\mathrm{E}}{a\sqrt{\frac{b^2}{a^2}+\frac{c^2}{a^2}+\zeta^2+\frac{2c}{a}\zeta}}.$$

Choose two quantities η and ν, independent of the co-ordinates, such that

$$1-\nu^2+\eta^2 = \frac{b^2+c^2}{a^2}, \qquad \nu\eta = \frac{c}{a},$$

and ν^2 is less than unity, while η^2 is positive. The conditions necessary for these restrictions require a little discussion, and corresponding values of ν^2 and η^2 are

$$\nu^2 = \frac{1}{2a^2}\{\pm\sqrt{(b^2+c^2-a^2)^2+4a^2c^2}-(b^2+c^2-a^2)\}$$

$$\eta^2 = \frac{1}{2a^2}\{\pm\sqrt{(b^2+c^2-a^2)^2+4a^2c^2}+(b^2+c^2-a^2)\}.$$

Only one value of ν^2 can be positive, and from the equation for ν^2, namely,

$$a^2\nu^4+\nu^2(b^2+c^2-a^2)-c^2 = 0,$$

we see, by DESCARTES' rule, that it is between zero and unity in all circumstances. Thus

$$\nu = \frac{1}{a\sqrt{2}}\sqrt{\{(b^2+c^2-a^2)^2+4a^2c^2\}^{\frac{1}{2}}-(b^2+c^2-a^2)}$$

$$\eta = \frac{1}{a\sqrt{2}}\sqrt{\{(b^2+c^2-a^2)^2+4a^2c^2\}^{\frac{1}{2}}+b^2+c^2-a^2},$$

and η is evidently always real. With these values

$$v \quad \frac{\mathrm{E}}{a\sqrt{1-\nu^2+\eta^2+\zeta^2+(2c/a)\nu\zeta}} = \frac{\mathrm{E}}{a}\sum_0^\infty(-)^n(2n+1)\,q_n(\eta)\,\mathrm{P}_n(\nu)\,p_n(\zeta) \quad (\zeta<\eta)$$

$$= \frac{\mathrm{E}}{a}\sum_0^\infty(-)^n(2n+1)\,p_n(\eta)\,\mathrm{P}_n(\nu)\,q_n(\zeta) \quad (\zeta>\eta),$$

where η and ν are constants, and ζ is a spheroidal co-ordinate.

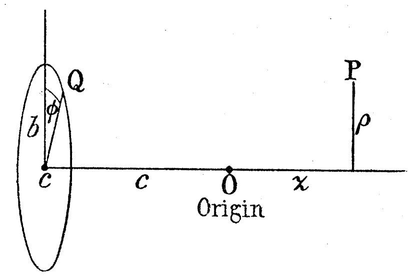
This is true on the axis of z, or $\mu = 1$, where μ is the other co-ordinate.

The appropriate generalisation to all points of space is

$$\mathrm{V} = \frac{\mathrm{E}}{a}\sum_0^\infty(-)^n(2n+1)\,q_n(\eta)\,\mathrm{P}_n(\nu)\,\mathrm{P}_n(\mu)\,p_n(\zeta) \quad (\zeta<\eta)$$

$$= \frac{\mathrm{E}}{a}\sum_0^\infty(-)^n(2n+1)\,p_n(\eta)\,\mathrm{P}_n(\nu)\,\mathrm{P}_n(\mu)\,q_n(\zeta) \quad (\zeta>\eta),$$

the series being necessarily convergent. These series are new, and it is of interest to note the corollary that their sum has been proved indirectly to be an elliptic integral. For, in the figure, which requires no explanation, we see that the potential at a point P of such a ring is



$$V = \int_0^{2\pi} \frac{E}{2\pi} \frac{d\phi}{PQ}$$

$$= 2\int_0^{\pi} \frac{E}{2\pi} \frac{d\phi}{\{(\rho - b\cos\phi)^2 + (b\sin\phi)^2 + (z+c)^2\}^{\frac{1}{2}}},$$

or

$$V = \frac{E}{\pi}\int_0^{\pi} \frac{d\phi}{\{(z+c)^2 + b^2 + \rho^2 - 2b\rho\cos\phi\}^{\frac{1}{2}}}$$

$$= \frac{E}{\pi}\int_0^{\pi} \frac{d\phi}{[c^2 + 2ac\mu\zeta + a^2\mu^2\zeta^2 + b^2 + a^2(1-\mu^2)(1+\zeta^2) - 2a^2\cos\phi\{(1-\mu^2)(1+\zeta^2)\}^{\frac{1}{2}}]^{\frac{1}{2}}}$$

$$= \frac{E}{a\pi}\int_0^{\pi} \frac{d\phi}{\left[1-\mu^2+\zeta^2+(b^2+c^2)/a^2 + \frac{2c}{a}\mu\zeta - \frac{2b}{a}\{(1-\mu^2)(1-\zeta^2)\}^{\frac{1}{2}}\cos\phi\right]^{\frac{1}{2}}}$$

$$= \frac{2E}{\pi}\left(\overline{z+c}^2 + \overline{b+\rho}^2\right)^{-\frac{1}{2}} K\left[\frac{2(b\rho)^{\frac{1}{2}}}{\{(z+c)^2+(b+\rho)^2\}^{\frac{1}{2}}}\right]$$

$$= \frac{2E}{a\rho}\{2-\mu^2-\nu^2+\eta^2+\zeta^2+2\mu\nu\eta\zeta+2\sqrt{1-\mu^2.\,1-\nu^2.\,1+\eta^2.\,1+\zeta^2}\}^{-\frac{1}{2}}$$

$$\times K\left[\frac{\{(1-\mu^2)(1-\nu^2)(1+\eta^2)(1+\zeta)\}^{\frac{1}{2}}}{\{2-\mu^2-\nu^2+\eta^2+\zeta^2+2\mu\nu\eta\zeta+2(1-\mu^2.\,1-\nu^2.\,1+\eta^2.\,1+\zeta^2)\}^{\frac{1}{2}}}\right]^{\frac{1}{2}}.$$

We thus find, in symmetrical form as regards η and ζ, or μ and ν, that if μ and ν do not exceed unity, and if η and ζ are real, while

$$\omega^2 = 2-\mu^2-\nu^2+\eta^2+\zeta^2+2\mu\nu\eta\zeta+2\sqrt{\{(1-\mu^2)(1-\nu^2)(1+\eta^2)(1+\zeta^2)\}}.$$

Then, if $\eta > \zeta$,

$$\sum_0^{\infty}(-)^n(2n+1)\,q_n(\eta)\,P_n(\nu)\,P_n(\mu)\,p_n(\zeta) = \frac{2}{\pi\omega}K[\omega^{-1}\sqrt{\{(1-\mu^2)(1-\nu^2)(1+\eta^2)(1+\zeta^2)\}^{\frac{1}{2}}}],$$

and if $\eta < \zeta$,

$$\sum_0^{\infty}(-)^n(2n+1)\,q_n(\zeta)\,P_n(\nu)\,P_n(\mu)\,p_n(\eta) = \frac{2}{\pi\omega}K[\omega^{-1}\{(1-\mu^2)(1-\nu^2)(1+\eta^2)(1+\zeta^2)\}^{\frac{1}{2}}].$$

These formulæ have also been proved otherwise, but an alternative proof is not regarded as necessary in this memoir.

When μ or ν become equal to unity, formulæ already obtained in previous sections are found as special cases.

§24. *Some Analytical Formulæ.*

The special case $\mu=0$ is of some importance. The right-hand side becomes an even function of the other variables, and

$$\omega^2 = 2-\nu^2+\eta^2+\zeta^2+2\{(1-\nu^2)(1+\eta^2)(1+\zeta^2)\}^{\frac{1}{2}}.$$

With this value, our formulæ become

$$\sum_0^\infty (-)^{2n}(4n+1)\, q_{2n}(\eta)\, P_{2n}(\nu)\, p_{2n}(\zeta)\, P_{2n}(0) = \frac{2}{\pi\omega} K[\omega^{-1}\{(1-\nu^2)(1+\eta^2)(1+\zeta^2)\}^{\frac{1}{2}}] \quad (\eta > \zeta),$$

and a similar one when $\eta < \zeta$.

The restriction of ζ to real values can also be removed, though we include no formal proof, the lines of its construction being fairly obvious. With $\zeta = \iota\gamma$, $p_{2n}(\zeta) = (-)^n P_{2n}(\gamma)$, and thus

$$\sum_0^\infty (-)^n (4n+1)\, q_{2n}(\eta)\, P_{2n}(\nu)\, P_{2n}(\gamma)\, P_{2n}(0) = \frac{2}{\pi\omega} K[\omega^{-1}\{1-\nu^2 \,.\, 1-\gamma^2 \,.\, 1+\eta^2\}^{\frac{1}{2}}],$$

where now

$$\omega^2 = 2-\nu^2-\gamma^2+\eta^2+2\{1-\nu^2 \,.\, 1-\gamma^2 \,.\, 1+\eta^2\}^{\frac{1}{2}},$$

and ω is real if γ is between ± 1.

Multiplying both sides by x/γ^2+x^2 and integrating between zero and unity,

$$\sum_0^\infty (4n+1)\, q_{2n}(\eta)\, q_{2n}(x)\, P_{2n}(\nu)\, P_{2n}(0) = \frac{2x}{\pi}\int_0^1 \frac{d\gamma}{x^2+\gamma^2} \cdot \frac{1}{\omega} K[\omega^{-1}\{1-\nu^2 \,.\, 1-\gamma^2 \,.\, 1+\eta^2\}^{\frac{1}{2}}],$$

with the value of ω last written.

The sum of this series is required in the expression of the potential due to a disc at zero potential in presence of a parallel and coaxial electrified ring.

§25. *A Thin Electrified Ring is Coaxial with an Oblate Spheroid. To Find the Charge Induced on the Spheroid when it is at Zero Potential.*

If the spheroid is

$$\frac{z^2}{a^2\zeta_0^2} + \frac{\rho^2}{a^2(1+\zeta_0^2)} = 1,$$

and the ring $z = -c$, $\rho = b$, with charge E, the potential of the ring alone is

$$V = \frac{E}{a}\sum_0^\infty (-)^n (2n+1)\, q_n(\eta)\, P_n(\nu)\, P_n(\mu)\, p_n(\zeta)$$

near the surface of the spheroid, where

$$\nu = \frac{1}{a}\sqrt{2} \,.\, [\{(b^2+c^2-a^2)^2+4a^2c^2\}^{\frac{1}{2}} - (b^2+c^2-a^2)]^{\frac{1}{2}}$$

$$\eta = \frac{1}{a}\sqrt{2} \,.\, [\{(b^2+c^2-a^2)^2+4a^2c^2\}^{\frac{1}{2}} + (b^2+c^2-a^2)]^{\frac{1}{2}}.$$

The potential due to the charge on the spheroid, where K_n is constant, is

$$V' = \frac{E}{a}\sum_0^\infty (-)^n (2n+1)\, q_n(\eta)\, P_n(\nu)\, P_n(\mu) \,.\, K_n q_n(\zeta),$$

and as $V+V' = 0$ when $\zeta = \zeta_0$, for all values of μ, $K_n = -p_n(\zeta_0)/q_n(\zeta_0)$.

Thus

$$V' = -\frac{E}{a}\sum_{0}^{\infty}(-)^n(2n+1)\,q_n(\eta)\,P_n(\nu)\,P_n(\mu)\,q_n(\zeta)\,p_n(\zeta_0)/q_n(\zeta_0).$$

At a great distance, the first term, or the term in $1/r$, is

$$V' = -\frac{E}{a}q_0(\eta)\,q_0(\zeta)\,p_0(\zeta_0)/q_0(\zeta_0) = -\left[\frac{E}{a}\cdot q_0(\eta)/q_0(\zeta_0)\right]\frac{a}{r},$$

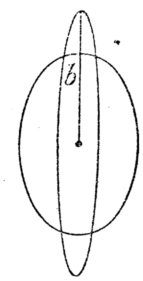
so that the charge induced on the spheroid is

$$E' = -Eq_0(\eta)/q_0(\zeta_0) = -\frac{E}{\sin^{-1}e}\cot^{-1}\eta$$

where e is the eccentricity of the meridian section of the spheroid. If the semiaxes are A and B, with $A > B$, $a^2 = A^2 - B^2$, and

$$E' = -\frac{E}{\sin^{-1}e}\cot^{-1}\sqrt{\left\{\frac{\{(b^2+c^2-A^2+B^2)^2+4c^2(A^2-B^2)\}^{\frac{1}{2}}+(b^2+c^2-A^2+B^2)}{2(A^2-B^2)}\right\}}.$$

This formula is new. If the ring encircles the spheroid, as in the figure, $c = 0$, and



$$E' = -\frac{E}{\sin^{-1}e}\cot^{-1}\left(\frac{b^2-A^2+B^2}{A^2-B^2}\right)^{\frac{1}{2}} = -\frac{E}{\sin^{-1}e}\sin^{-1}\left(\frac{\sqrt{A^2-B^2}}{b}\right),$$

a remarkably simple formula.

For the special case of a circular disc, of radius A, $B = 0$, and $c = 1$, so that

$$E' = -\frac{2E}{\pi}\cdot\cot^{-1}\sqrt{\left\{\frac{[(b^2+c^2-A^2)^2+(4A^2c^2)]^{\frac{1}{2}}+(b^2+c^2-A^2)}{2A^2}\right\}}.$$

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